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THE FOUNDATIONS
OF ALTERNATE CURRENT THEORY

GENERAL

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OF

ALTERNATE CURRENT THEORY

BY

C. V. DRYSDALE

D.Sc. (Lond.), M.I.E.E.

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PREFACE

THE present volume is a *résumé* of the theoretical portion of a course on Alternate Current Working which the Author has given for the past five years at the Northampton Institute. The guiding principles in these lectures have been the reduction of the mathematical work to a minimum, and the basing of the theory upon simple mechanical analogies, which have been found to appeal strongly to practical engineers. Most teachers of the subject must at one time or another have employed mechanical illustrations of the results of the theory, and several text-books contain such illustrations; but it is believed that this is the first attempt to found a complete and logical theory upon a mechanical basis. The inspiration was derived from the ether models of Maxwell and Lodge, and some of the models here described are obviously modifications of them.

It will be observed that by the aid of these analogies and models, the most important parts of alternate current theory can be mastered with no more mathematical knowledge than is involved in the handling of simple equations and right-angled triangles; and it is believed that the physical conception of the processes and results are of the greatest value even to the most facile mathematician. As will be shown, it is hoped, in a future volume, many of the more advanced problems can be similarly simplified, but it is not in any way suggested that this method of treatment should be exclusively followed, to the

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detriment of mathematical study. On the contrary, it is hoped that it will serve as an introduction to the excellent works of Dr. Hay, Professor S. P. Thompson, Dr. Alexander Russell, Professor Steinmetz, and others, and that it will stimulate the interest of students in mathematical processes and results. On the other hand, the author has the greatest belief in Mr. Heaviside's remark that 'the best result of mathematics is to be able to do without it,' and in the French proverb, '*On ne peut pas dire qu'on a le dernier mot d'une théorie, jusqu'on peut l'exprimer en peu de mots à un passant dans la rue.*' The most unfortunate feature of engineering in this country is the conflict between 'theory and practice,' and a greater popular appreciation of science can only be obtained by simplifying it as much as possible.

The use of British rather than of C.G.S. units may seem a retrograde step at the present time, but has been adopted after lengthy consideration. As the chief aim of the book is to divest the subject of its unfamiliarity and to enable the phenomena to be visualised, it would be inconsistent to appeal to grammes, centimetres, and dynes until it has become a habit to deal with such quantities in daily life.

It will, of course, be obvious that the mode of treating the subject here developed lends itself excellently to practical laboratory work, and in many cases with very simple apparatus.* The Author, some years ago, projected a special laboratory course in which the analogous mechanical and electrical experiments would be arranged side by side; and in which a number of moving vector diagrams or linkages would be available. Unfortunately, facilities have not been available for carrying out this idea, but the Author believes that such a course could be made of the greatest interest and value to students with comparatively small trouble and expense, and that it would even apply to testing electrical machinery. A perfect no-load and short-circuit test could, for example, be performed on the mechanical transformer model of p. 212.

---

* The majority of the models described in the text can be obtained if required from Messrs. G. Cussons, Ltd., Scientific Instrument Makers, Manchester.

The Author's thanks are gratefully tendered to his colleagues at the Northampton Institute, who have given great help in the production of the book. Mr. L. W. Phillips and Mr. A. F. Burgess, B.Sc., are responsible for most of the diagrams and numerical calculations; Mr. T. B. Baker has made up the models and most of the photographs; and the Poulsen arc and oscillatory discharge experiments were made by Mr. C. M. Dowse. The whole of the proofs have been kindly revised by Mr. A. C. Jolley, who has also made many useful suggestions.

It is hoped that no important errors have escaped notice, but the Author would be grateful if anyone finding mistakes would communicate them to him.

C. V. D.

REIGATE, *March* 1910.

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LIST OF PRINCIPAL SYMBOLS AND CONVENTIONS

F	Force in lbs.
s	Displacement in ft.
v	Velocity in ft. per sec.
a	Acceleration in ft. per sec. per sec.
Y	Yield Constant in ft. per lb.
f	Friction Constant in lbs. per ft. per sec.
m	Mass in Engineers' units.
e	Energy in ft.-lbs.
P	Power in ft.-lbs. per sec.

V	P.D. in volts.
Q	Quantity in coulombs.
C	Current in amperes.
A	Electrical Acceleration in amperes per sec.
K	Capacity in farads.
r	Resistance in ohms.
L	Inductance in henries.
J	Energy in joules.
w	Power in watts.
M	Mutual inductance in henries.

F, s, v, V, Q, C, &c.	Instantaneous values.
$\widehat{\mathrm{F}}$, $\hat{s}$, $\hat{v}$, $\hat{\mathrm{V}}$, $\hat{\mathrm{Q}}$, $\hat{\mathrm{C}}$, &c.	Maximum values.
$\bar{\mathrm{F}}$, $\bar{s}$, $\bar{v}$, $\bar{\mathrm{V}}$, $\bar{\mathrm{Q}}$, $\bar{\mathrm{C}}$, &c.	Effective values.
$\hat{\mathrm{F}}$, $\hat{s}$, $\hat{v}$, $\hat{\mathrm{V}}$, $\hat{\mathrm{Q}}$, $\hat{\mathrm{C}}$, &c.	Vector or quaternion effective values.

F_1 or v_1	Force or velocity for spring.
F_2 or v_2	Force or velocity for friction.
F_3 or v_3	Force or velocity for mass.
F_x or v_x	Horizontal component of force or velocity.
F_y or v_y	Vertical component of force or velocity.
I	Impedance in ohms.
Y	Admittance in mhos.
x	Reactance in ohms.
g	Conductance in mhos.
b	Susceptance in mhos.
$\angle\phi$	Leading by angle ϕ.
n	Frequency ~ per second.

V_1 or C_1	P.D. or current for Condenser.
V_2 or C_2	P.D. or current for Resistance.
V_3 or C_3	P.D. or current for Inductance.
V_x or C_x	Horizontal component of P.D. or current.
V_y or C_y	Vertical component of P.D. or current.
$\hat{\mathrm{I}}$	Impedance, as vector.
$\hat{\mathrm{Y}}$	Admittance, ,, ,,
V_a or C_a	Subscript letters for different circuits.
V_b or C_b	
$\ulcorner\phi$	Lagging by angle ϕ.
$p =$	$2\pi n$.

Conventions in Diagrams containing Curves

Total Force or P.D.	thin line.
,, Velocity or Current	thick line.
Component for Spring or Condenser	————————
,, ,, Friction or Resistance	-- -- -- --
,, ,, Inertia or Inductance	--- --- --- ---

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THE FOUNDATIONS OF ALTERNATE CURRENT THEORY

PART I

FUNDAMENTAL MECHANICAL AND ELECTRICAL RELATIONS

CHAPTER I

CONTINUOUS CURRENTS

NUMEROUS as may be the applications of continuous currents, and complicated as may be the batteries and dynamos by which they are produced, their theory is in reality extremely simple, so far as the calculation of the strength of currents is concerned. All that is really necessary is summed up in the two laws of Ohm and of Kirchhoff, the former of which states that the P.D. between any two points of a circuit is proportional to the current, and to the resistance of the circuit between those points; while the latter asserts that current cannot be accumulated at a point, and that if any number of circuits join at a point, the total current flowing away from it will equal the total current flowing towards it.

Now what do these statements teach us as to the nature of the electric current? We know from the simplest facts concerning it that the electric current may be compared to a flow of water, and in fact the very word 'current' implies that we think of it as a flow of some fluid. We may put side by side the comparison between the electric current and a stream or current of water in a pipe as follows:

WATER CURRENT	ELECTRIC CURRENT
1. The water requires a perfect circuit of pipes in order to flow continuously. If stopped by a valve at any point, its motion ceases at all points of its course.	1. The current requires a perfect circuit of conductors in order to flow continuously. If stopped by a break at any point, it ceases at all points of the circuit.
2. The quantity of water which passes each portion of the pipe in unit time is the same, since the water is incompressible.	2. The strength of the current at all parts of a simple circuit is the same.
3. The water appears to have a definite direction of flow, as evidenced by the motion of particles carried along with it.	3. The current appears to have a definite direction of flow, as evidenced by its behaviour on a magnetic needle in its neighbourhood. If the current is reversed the needle deflects in the opposite direction.
4. The water requires an impulse or Water-Motive Force to keep it in motion.	4. The current requires a voltage or Electro-Motive Force to keep it flowing.
5. The flow of the water is resisted by the friction of the pipe; which is greater the longer the travel, the narrower the bore, and the rougher the surface of the pipe.	5. The flow of the current is resisted by the resistance of the circuit; which is greater the longer the conductor, the smaller its section, and the higher its specific resistance.
6. The forcing of the water to flow against the friction of the pipes entails an expenditure of energy which is converted into heat.	6. The forcing of the current against the resistance of the conductor entails an expenditure of energy which is converted into heat.
7. When water is flowing along a pipe, a difference of pressure exists between any two points, depending upon the rate of flow of the water and the amount of friction against the walls of the pipe.	7. When a current flows through a conductor, a difference of potential exists between any two points, depending upon the strength of the current and the resistance of the conductor.

8. If any number of pipes meet in a point, the total flow away from that point must equal the total flow towards it.—*Law of continuity.*	8. If any number of conductors meet in a point, the total current from that point must equal the total current towards it.—*Kirchhoff's law.*

These comparisons are quite enough to show that there is a very strong analogy, if nothing more, between what we call an electric current, and the flow of an incompressible liquid such as water. We say incompressible, because if this were not the case the flow could continue in some parts of a pipe when stopped at another, or could have different values at different parts of the circuit, and Kirchhoff's law that the current cannot accumulate at a point would not find its hydraulic analogy.

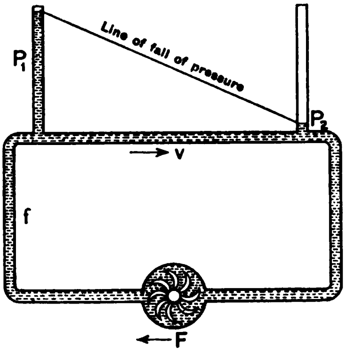

FIG. 1.—Water circuit.

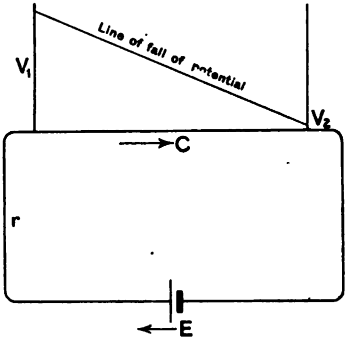

FIG. 2.—Electric circuit.

The comparison is diagrammatically illustrated in Figs. 1 and 2, where the pump* and battery provide the Water-Motive Force *F* and the Electro-Motive Force *E*, producing the velocity of flow *v* and the current *C*, against the friction *f* and the resistance *r* in each case respectively.

Before going further, we may notice that the movement of any object such as a tramcar or boat has a certain resemblance to an electric current. All such motions have a definite direction, they require a driving force to maintain them, being resisted by friction, and when the body is driven against this friction they absorb energy and convert it into heat. In each case the driving force *F* corresponds to an E.M.F. *E*, and the velocity of motion *v* to the current *C*, which might be spoken of as electrical velocity.

---

* The pump in the diagram is purely conventional and is intended to represent any form of centrifugal pump.

Now comes the question: What is Ohm's law? It is a quantitative statement which gives us the relation between the E.M.F. required in a circuit to produce a certain current or electrical velocity, and that current. Suppose that we have a body such as a tramcar on level rails. It will be found that a certain force is required to start it moving, and that the same force will keep it moving at almost any speed, so long as it is not so great as to cause much wind resistance. For the force of friction between the surfaces of solid bodies is approximately independent of speed over a considerable range.

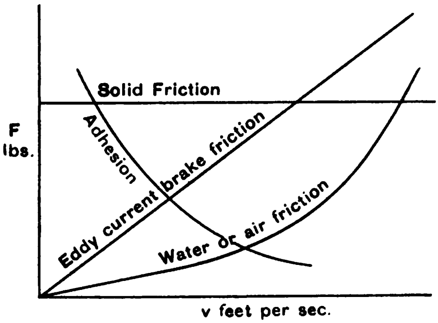

FIG. 3.—Friction curves.

We may represent this graphically in Fig. 3, in which the velocity v of the body is plotted horizontally, and the driving force F vertically. In the case just given the force-velocity curve should be a horizontal straight line, showing that the force is the same for all speeds. When we have some adhesive substance between the body and the surface over which it slides, the force even gets smaller as the speed increases. If, on the other hand, we take the case of water flowing in the pipe, or of a ship in still water, we find that an infinitesimally small force is sufficient to just move it, and that the force must be increased in order to make the motion more rapid. If, for example, a force of f pounds was found necessary to produce a speed of 1 ft. per sec., then a speed of 2 ft. per sec. would be obtained with a force of approximately $2f$, 3 ft. per sec. with $3f$ lbs., and so on. If this is plotted we have, of course, a straight line through the origin expressing the proportion of F to v. But on endeavouring to increase the velocity beyond five or six ft. per sec. it is found that eddies are formed, and the force has then to be increased very much more rapidly than in proportion to the speed. This is the reason for the disproportionately large power required in the engines of a 20-knot boat compared with the power for 15 knots. In the force-velocity curve this is represented by the turning up of the curve at high speeds, and the force, instead of being proportional to the speed as we found at low speeds, is proportional to the square of the speed. A similar curve would represent air friction.

There is one kind of friction, however, in which the proportionality of force to velocity holds over almost any range. If we take a plate of copper and hold it between the poles of a powerful electromagnet, no force is felt so long as the metal is stationary, but on moving it across the field we experience a resistance. It is found in this case that the resisting force is exactly proportional to the speed over any range until the plate begins to heat appreciably, and this proportionality is turned to account in the brake discs of electric meters.

We therefore see that there is no one law expressing the relation of driving force to velocity in the case of a moving body. If, however, we take the case of water or air friction at low speeds or eddy current friction, the law of friction may be stated as—force is proportional to velocity or $F \propto v$. Now suppose we have a body which moves according to this law, and that we attach a spring balance to it and pull it forward at a speed of 1 ft. per sec. The spring balance will then register a pull which we will denote by f lbs. Since the force is proportional to the velocity, we should require a force of $2f$ lbs. to pull the body at 2 ft. per sec. and of vf lbs. to pull it with a velocity of v ft. per sec.; i.e. $F = fv$ where F is the driving force in lbs., v the velocity in feet per second, and f is the force in lbs. which is required to move the body at a speed of 1 ft. per sec. The latter force f we shall term the *friction-constant* of the body. This law is only true for the particular kind of friction we have been dealing with, but would be fairly closely correct in the case of a boat moving at slow speeds.

Now let us return to the electrical circuit. Suppose that we have a resistance r connected with an ammeter and electrostatic voltmeter as in Fig 4. We can cause different currents to flow through the resistance, and in each case note the voltage required to cause this current. The result may be plotted as a curve in Fig. 5.

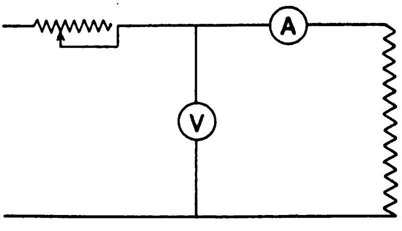

FIG. 4.—Testing the characteristic of a resistance.

In the majority of cases it will be found that increase of the current C is attended with increase of the voltage V. With a resistance made of a couple of similar plates in an acid solution, however, the P.D. will be almost constant whatever the current, if the plates are large and near together, owing to the back E.M.F. of polarisation. This gives a horizontal line, practically corresponding to that for solid friction. If the circuit is a loop of ordinary wire the P.D. will at first rise in proportion to C, but as soon as the wire gets hot it rises much more rapidly. In the case of the bolstering resistance of the Nernst lamp this is extremely marked. On the other hand, with a liquid resistance composed of small plates at a distance apart, or a carbon resistance, the rise of voltage is less rapid as the temperature rises, while in the case of an arc the P.D. actually falls as the current is increased, corresponding to the case of mechanical adhesion. A curve of this kind is sometimes called the 'characteristic' of a resistance. If, therefore, the current is not so great as to cause appreciable heating, or if the wire is kept at the same temperature artificially, or, lastly, if the wire is made of a special alloy, the characteristic is a straight line and $V \propto C$. If we then find out how many volts are required to force one ampere through the coil and denote this by r, we shall have $V = rC$ as the voltage required in other cases. The quantity r is called the *resistance* of the conductor in ohms, and it corresponds exactly to what we called the 'friction-constant' in the case of the moving body, as it is the electrical force in volts required to produce a unit electrical velocity of one ampere.

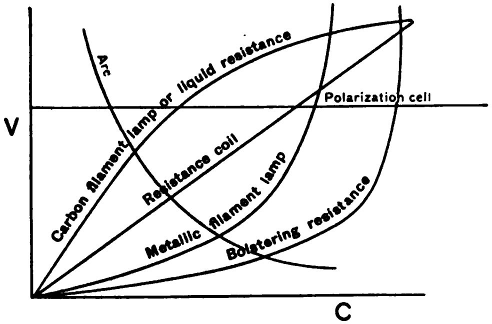

FIG. 5.—Resistance characteristics.

The conclusions that we have come to may therefore be summarised as follows:

(*a*) Kirchhoff's law holds equally for the movement of water currents in pipes or of electric currents in circuits.

(*b*) For all cases where the force required to move a body is proportional to the speed, we have the relations—

$$F = fv, \text{ or } v = \frac{F}{f} \quad . \quad . \quad . \quad . \quad . \qquad (1a)$$

In the case of an electrical circuit of constant resistance, we have the corresponding relations—

$$V = rC, \text{ or } C = \frac{V}{r} \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

The real significance of Ohm's law is therefore simply that the electrical force is exactly proportional to the electrical speed or current.

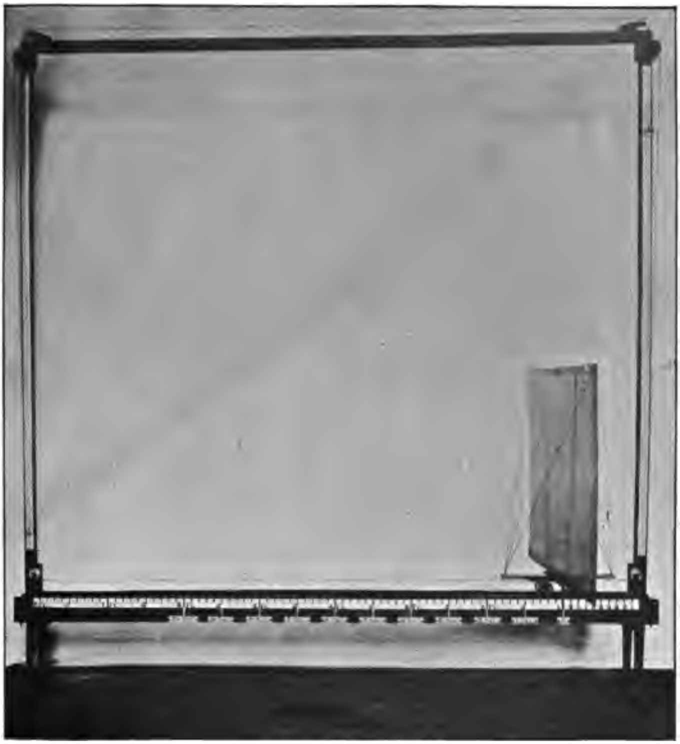

FIG. 6.—Mechanical model of circuit having resistance.

The behaviour of the electric circuit can be fairly closely imitated by a model such as that shown in Fig. 6. It consists of a wooden frame about 8 ft. square, with pulleys at the four corners round which a cord passes representing a closed electric circuit. On the bottom of the frame two rails are laid, upon which runs a light aluminium truck with axles in ball-bearings; and this truck is attached to the cord. A force can be applied to the circuit by putting a weight on a scale-pan attached to one side of the cord, and if a large sail or a paddle, moving in a water trough, is fixed to the truck, its motion will be retarded by fluid friction. There will, of course, be a small amount of solid friction as well in the pulleys and truck, but we can compensate for this by putting weights in the scale-pan until the truck will just crawl along without getting up any appreciable speed. If now an additional weight of, say, 1 oz. is added, it is found that the truck starts and moves with a steady speed of, say, 6″ per sec. This can be readily shown (see Fig. 6) by having a seconds pendulum by the side and making a chalk mark on the scale opposite the pointer on the truck at every tick. With a weight of 2 ozs. the marks are found to be 1 ft. apart, or the speed is twice that produced with the 1 oz. weight, and so on. In this case, since the 2 oz. weight produces a speed of 1 ft. per sec., the friction-constant of the sail is ·125 lb. per ft. per sec., and the speed produced by a weight or force of any amount will be given by dividing the weight in pounds by this friction-constant.

So far, beyond having presented the ordinary notions of continuous currents in a somewhat novel light, we do not seem to have gained much by the analogy. We have now, however, to go a stage further, and the reason for having employed it will be apparent.

CHAPTER II

INTERMITTENT MOTION, INERTIA, AND INDUCTANCE

In our last chapter the analogy was drawn between the circulation of water in pipes, the motion of a boat or tramcar, and the passage of an electric current. We saw that Kirchhoff's law simply expressed the ordinary continuity of flow in an incompressible fluid, and that Ohm's law is equivalent to a statement that the force of electrical friction is proportional to the rate of flow. If the motion is steady, these two laws are sufficient to determine the flow or motion in all cases.

But now consider the water or tramcar again, and suppose that, instead of its moving steadily under the action of a uniform force, it is at rest and the force is suddenly applied. Will the water or car start instantly with its full velocity? We know that it will not. When a tramcar is to be started, the horses may be seen pulling the traces taut, and the car only gets up speed slowly; while when it is once up to speed on the level, they are trotting along freely, exerting hardly any effort to keep it going. Again, when the car is in motion and the horses suddenly stop pulling, the motion does not cease instantly, but only by degrees, and a considerable retarding force has to be provided by brakes to stop it quickly.

Evidently, therefore, there is something else besides friction which influences the motion of a body when starting or stopping, and that something is the mass or inertia of the body. In the case of our water circuit, if the pump were suddenly started so as to give a definite pressure, the water current would only start gradually, and would rise by degrees to its full velocity; while, if the force were suddenly removed, the water would continue to circulate for some time before entirely coming to rest.

Another thing to notice is that, when any heavy body has got up speed, the force required to stop it depends greatly upon the time taken to do so. If this time is very short, then the force must be enormously great. If a hammer has a high velocity before striking a nail, it exerts a force on the head of the nail many times greater than that which could be obtained by the pressure of the hand. If a heavy car running at a high speed is suddenly checked by an obstacle, it may crash through it. Similarly, if water flowing at a high speed in a long pipe is suddenly stopped by a tap, a shock like a hammer blow is heard; and this shock may be even enough to burst the pipe, while the ordinary water pressure is comparatively small.

Is there anything in the electric circuit which corresponds to the inertia of the water or body? If the analogy which we drew in the last chapter is complete, we should expect this to be the case.

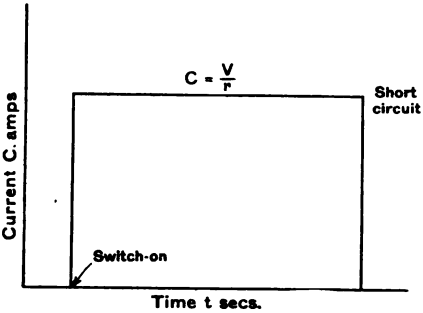

FIG. 7.—Rise and fall of current in a resistance.

Now in many cases we may quite fail to notice any such effect. If we switch on a resistance to mains at a steady voltage, the ammeter needle flies at once to a certain reading, and remains there steadily, providing that it does not swing to-and-fro once or twice owing to its own inertia. The better the instrument, the more quickly will it reach a steady reading, and there is nothing to show that the current has not risen instantaneously. Indeed, if Ohm's law were strictly true, it

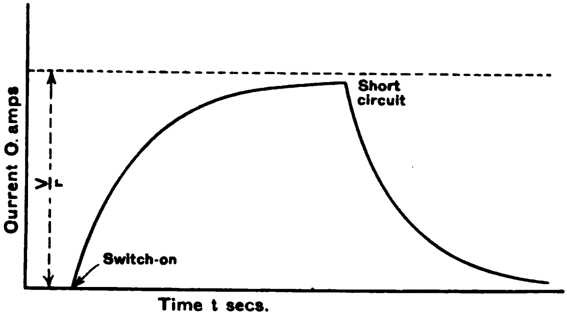

FIG. 8a.—Rise and fall of current in field coil.

would appear that this should be the case. If $C = \frac{V}{r}$, then the instant a force V is applied to a resistance r a current $C = \frac{V}{r}$ should flow, and remain steady, as shown in Fig. 7.

But cases are not unusual, although less frequent, when the current does not rise thus rapidly. If the exciting circuit of a large generator be switched on suddenly, even a poor ammeter will show that the current rises comparatively gradually, and it may take several seconds to reach its steady value. If a rapidly moving, dead-beat ammeter be used, it will be noticed that whenever the circuit contains a large electromagnet the current does not rise instantaneously. It rises, in fact, in some such way as that shown in Fig. 8*a*; Fig. 8*b* is the trace actually observed on switching on the field coils of a generator.

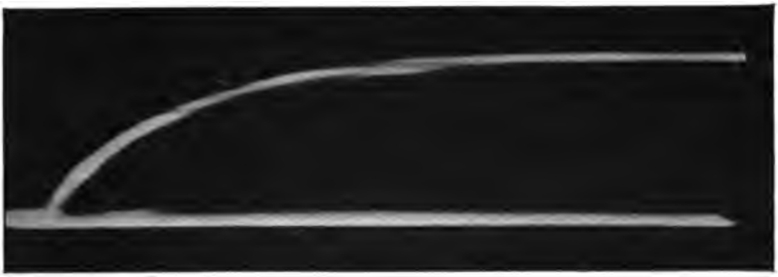

FIG. 8*b*.—Rise of current in field coil.

If when the current is flowing the voltage is suddenly removed, one might then expect to find the current running on like the tramcar or the water. But to remove the force it will not do to switch off, as this breaks the circuit, and is equivalent to interposing an obstacle in the path of the flow. But if the circuit and ammeter are connected, as in Fig. 9, with a fuse or circuit-breaker in the supply circuit, so that the mains may be short-circuited, then the closing of the short-circuiting switch is exactly equivalent to removing the force, and allowing a free path for the body. In this case it

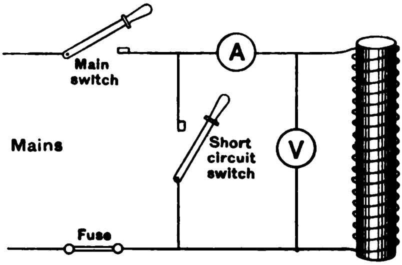

FIG. 9.—Method of discharging coil.

will be found that if the circuit is a simple resistance, the ammeter needle drops to zero immediately the short-circuit switch is closed, as in Fig. 7; but if this is done with the exciting coils or other electromagnet in circuit, the needle comes back slowly as in Fig. 8*a*. Both at starting and stopping, therefore, the electric current behaves as if it had inertia, when there is an electromagnet in circuit.

But the analogy holds even more fully when we actually break the circuit. As stated above, this is exactly equivalent to interposing an obstacle in the path of the moving body, since the air-gap formed by breaking the circuit is an insulator. The result of such a sudden check in the case of the tramcar or water was to produce a shock or crash which might break through the obstacle. Exactly the same thing is noticed with the current in an electromagnet. When a circuit which does not contain such a magnet is broken, the spark is comparatively small, and no arc will be formed unless the P.D. is high and the separation is gradual and small. But if the same current at the same voltage is passed through a large magnet, and the switch is opened, a brilliant sudden flash may be obtained which is immeasurably brighter than that obtained with the non-magnetic circuit (see Figs. 10 and 11), and the more rapidly the switch is opened the more intense is the flash. The tendency of the current to run on when once started is in this case so great as to cause it to leap across an air-gap many times greater than that which the pressure of the supply could possibly bridge.

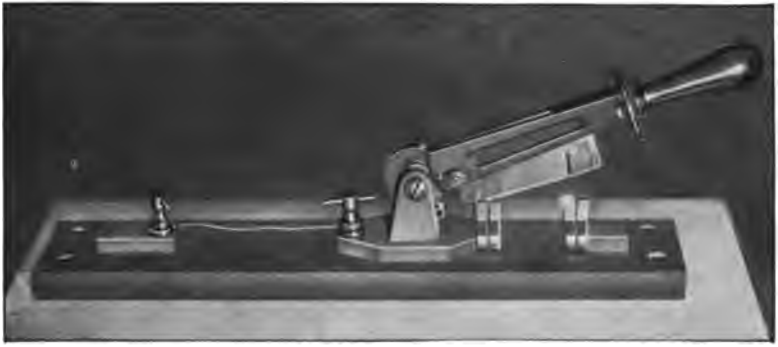

FIG. 10.—Spark on breaking non-inductive circuit (100 volts 2 amps.).

The voltage at the terminals of the coils at the moment when the current is being checked may, in fact, be as much greater than the normal voltage as the pressure of the hammer head on the nail at the moment of impact is greater than the force of the wielder's arm, or as the force at the moment of collision of a tramcar with an obstacle is greater than the driving force of the horses or motors.

The serious results of breaking the exciting circuit of a large field-magnet are well known. If such a magnet is connected to a 100-volt exciting circuit and the switch is suddenly broken, the impulse of the current to run on is so great that the P.D. across the coil may rise to many thousands of volts, and this frequently leads to a breakdown of the insulation of the coil.

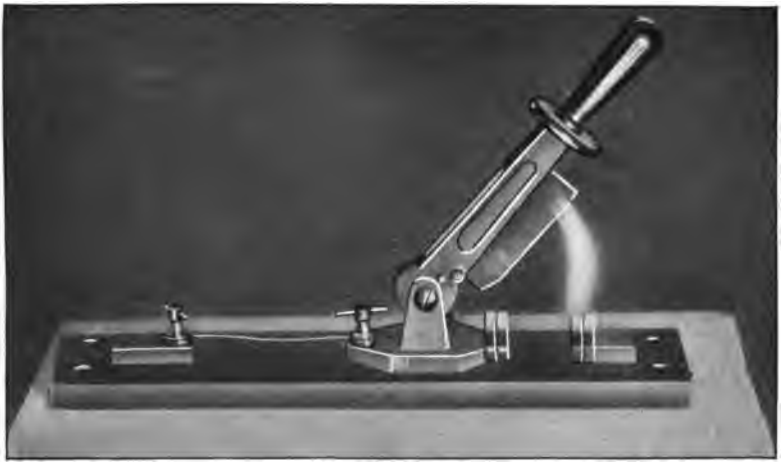

FIG. 11.—Spark on breaking inductive circuit (100 volts 2 amps.).

To avoid this the magnet must be shunted with a resistance before breaking, so as to give a by-pass through which the current may continue its circulation; the by-pass thus serves the same function as the net into which an acrobat dives, or the hydraulic buffers provided at railway termini. The rise of P.D. which is produced in this manner is vividly shown when a shunt generator, provided with a pilot lamp across its terminals, is running light, and then has one of its brushes lifted so as to break its excitation. If the machine is of any size, the lamp will flash up to a dazzling brightness, and will probably be broken, even if the speed of the machine is reduced to such an extent that the pilot lamp is at first glowing with only a dull red light.

These facts and experiments are amply sufficient to show that an electric circuit containing an electromagnet behaves like a moving body having inertia. To what extent does this analogy hold, and to what is it due? To answer this we must first state the effects of inertia in more definite form.

Law of Inertia.—Imagine a solid mass, as in Fig. 12, supported in such a way that there is absolutely no frictional resistance to its motion. Then we know that, if no force acts on it, it will either remain stationary or keep steadily moving in a straight line for ever. Of course this cannot be shown experimentally, as it is impossible to obtain a perfectly frictionless support, but we are absolutely certain of the truth of the statement from the fact that, if the motion alters in any way, we always find that there is a force acting on it of amount sufficient to produce the change. The motion of a body is therefore no proof that a force is acting on it, but a change of motion is. The definition of force, in fact, is 'that which tends to change the motion of a body.'

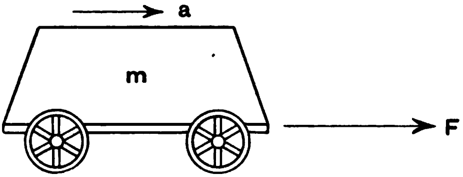

FIG. 12.—Force acting on mass.

Suppose, then, that we have such a frictionless mass, at rest, and we suddenly drag it with a steady force. The body will begin to move slowly at first like a railway train when starting, and will steadily gather speed. This increase of speed is called acceleration. Now it is found experimentally that, if the force is constant, the acceleration remains constant; that is, the velocity increases by equal amounts in each second. For instance, in the case of a weight allowed to fall freely under the pull of the earth, and consequently with a constant attractive force, we find that at the end of one second the speed of the body is 32·2 ft. per second, at the end of two seconds 64·4 ft. per second, at the end of three seconds 96·6 ft. per second, and so on, so long as the friction of the air does not affect it. Hence the speed increases by 32·2 ft. per second in each second, or the body is said to have an acceleration of 32·2 ft. per sec. per sec.

But if we suppose that our body had a mass of one pound, then the earth would be pulling it with a force of one pound. Hence we may say that a force of 1 lb. acting on a mass of 1 lb. gives it an acceleration of 32·2 ft. per sec. per sec.

Now it is clear that the force required to accelerate a body is proportional to its mass. This is shown experimentally by the fact that a 1-lb. weight and a 2-lb. weight, if let fall at the same moment from the same height, will reach the ground together. The 2-lb. weight has double the mass, but the pull of the earth on it is twice as great, and the acceleration therefore remains the same.

We can also prove experimentally in many ways that the force required to accelerate a mass is proportional to the amount of the acceleration. Combining these results, we find that the force required to accelerate a mass is proportional to the product of the mass and acceleration, and that a force of 1 lb. is required to accelerate a mass of 32·2 lbs. at the rate of 1 ft. per sec. per sec. If a mass of 32·2 lbs. is called the engineers' unit of mass, as has been done by Prof. Perry, then we can say that a force of 1 lb. is required to give a mass of one engineers' unit an acceleration of 1 ft. per sec. per sec. Consequently; if we reckon the mass *m* of a body in engineers'

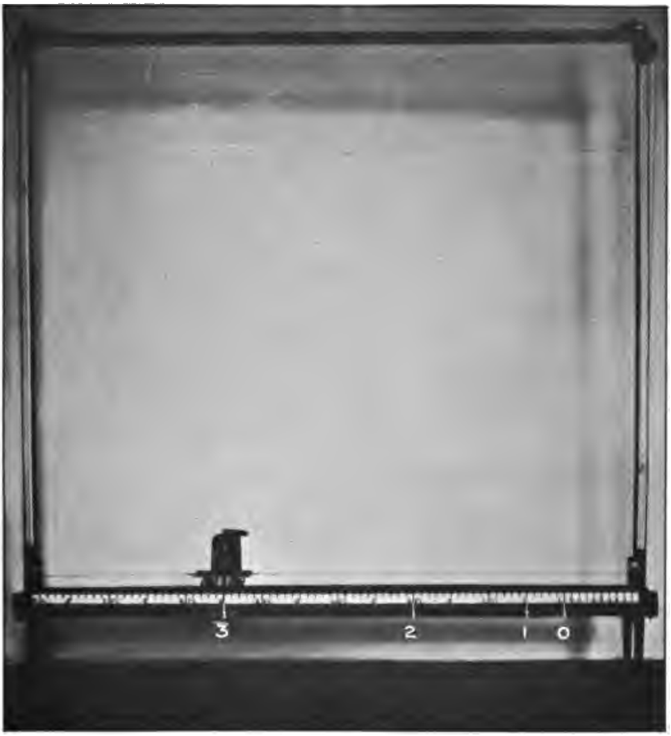

FIG. 13.—Mechanical model showing acceleration of mass.

units of 32·2 lbs. each and the acceleration a in feet per sec. per sec., the force F in lbs. required will be given by the relation—

$$F = ma \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

The truck model described on p. 7 serves also to prove roughly the foregoing statements. Suppose that we use the truck without the sail, and put weights on it until its total weight is 31 lbs., as in Fig. 13. We shall find that a small weight, say ¼ lb., will be necessary to compensate for the friction, *i.e.* to keep the truck moving steadily when once started. Now suppose that the truck is at rest, and that an additional weight of 1 lb. is suddenly added to the scale-pan. The truck will start at once, and will gradually get up speed, and if the seconds pendulum is at hand and the chalk marks are made every second as before, it will be found that the car has moved 6″ or ·5 foot in one second, 2 ft. in two seconds, and 4·5 ft. in three seconds. Now this means that in the first second the *average* speed of the truck = ·5 ft. per second. But the speed at the start was zero, hence the speed at the end of the second must have been 1 ft. per sec., if the increase of speed was uniform. Similarly in two seconds the total movement is 2 ft., giving an average speed of 1 ft. per second, and a speed at the end of the time of 2 ft. per sec. In 3 secs. the distance was 4·5 ft., the average speed being therefore $\frac{4{\cdot}5}{3} = 1{\cdot}5$ ft. per sec., and the final speed 3 ft. per sec. But we have here a force of 1 lb.* driving a total mass of 31 lbs. on the truck + 1¼ lbs. in the pan or 32·25 lbs., and we therefore come to the conclusion that a force 1 lb. causes a mass of 32·25 lbs. to attain a speed of 1 ft. per sec. in 1 sec., of 2 ft. per sec. in 2 secs., and of 3 ft. per sec. in 3 secs., and so on, or to accelerate at the rate of 1 ft. per sec. per sec. If the mass or force is varied in any way, it will be found that the acceleration in ft. per sec. per sec. produced is equal to the result obtained by dividing the force in lbs. by the mass in units of 32·2 lbs. This result could have been more accurately obtained by what is called an Attwood's machine, and in fact our truck model is simply a rough form of this machine, in which the mass moves horizontally instead of vertically.

Inductance.—Now let us go back to our circuit containing an electromagnet. In Fig. 14 we have a battery supplying current through a variable resistance and ammeter to an electromagnet. Suppose at first that no current is flowing. The core of the magnet is not magnetised, or only possesses a small and *steady* residual magnetisation. Now switch on the battery, and a current will begin to flow round the coil. But this current cannot flow without magnetising the core, and thus causing a number of lines of force to thread through the coil. In other words, the current in the coil cannot change without changing the magnetic flux or number of lines of force passing through it.

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* The extra ¼ lb. is not counted here, as its weight is expended in overcoming the friction of the truck, not in accelerating the mass.

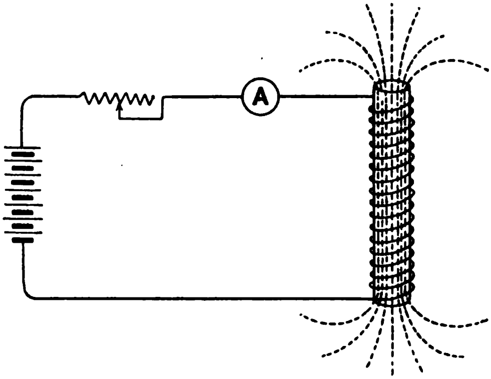

FIG. 14.—Explanation of electrical inertia.

But we know from Faraday's famous experiments in 1831 that whenever the number of magnetic lines passing through a circuit is changed, an E.M.F. is produced in that circuit. Consequently the current in the coil cannot change without an E.M.F. being produced. In order to settle the direction of this E.M.F. we have the law of Lenz, which states that the induced current always tends to stop the motion which produces it. This law was derived from experiments on moving magnets and coils of wire in such a way as to cause the number of lines of force threading the coil to be altered. Lenz's law therefore may be more generally expressed by saying that when the number of lines of force through a circuit is altered, the current produced by the induced E.M.F. always tends to stop the change of flux producing it. Since, in the case that we are considering, the change in the number of lines is produced by the change in the current round the coil, this means that the induced E.M.F. when the current changes is always in such a direction as to oppose the change of the current. If the current is growing stronger, the E.M.F. tries to keep it from so growing; and if it is dying away by the removal of the battery or otherwise, the induced E.M.F. tends to keep it flowing on. This is exactly what we have observed in our field-magnets, &c., and shows that the effect of this induction is to make the current behave as if it had inertia.

The induction in this case is called *self-induction* because it takes place in the exciting coil itself, instead of in a separate adjacent circuit, as in a transformer or induction coil, in which case it is called *mutual* induction. As self-induction is so important and is so much in evidence in all alternate-current matters, the term has been shortened to *inductance.*

Now what is the law of self-induction or inductance? How great is the induced E.M.F.? The answer is simple. Faraday not only proved that the change of the number of lines of force induced an E.M.F., but that this E.M.F. was proportional to the *rate* at which the number of lines were cut. It is from this that our unit of E.M.F., the *volt,* is derived.

One volt is the E.M.F. induced in a circuit when the number of lines of force passing through or linked with it changes at the rate of one hundred million, or 10^8, *per second.*

This is the *definition* of the volt, and consequently needs no proving. It is the basis of the formula which we use in obtaining the E.M.F. of an ordinary direct-current armature, which simply represents the rate at which lines of force are cut by the conductors, in hundreds of millions per second.

In the above definition it will be noticed that we have spoken of the lines of force as being linked with the circuit as well as of passing through it. The reason for this is that it is not only a question of the number of lines threading through the circuit, but of the number of times they do so. This will be quite clear from the two diagrams in Fig. 15. Suppose, as on the left (*a*), we have two permanent magnets, each carrying a flux of N lines, and that the circuit takes one turn round each. The total flux passing through the circuit is therefore $2N$ lines. But it is evident that there will be precisely the same inducing effect in the case shown on the right hand (*b*), where the two

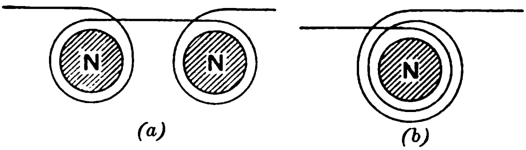

FIG. 15.—Explanation of inductance.

coils surround one magnet, in which the same flux N exists as in each of the magnets in (a), for on inserting or withdrawing the magnet there will be the same E.M.F. induced in each turn, and the total E.M.F. will thus be double of that for one turn. Hence, if a flux of N lines is encircled by n turns, it is equivalent in inductive effect to a total flux of Nn lines passing once through the circuit. When a line of force passes once through a circuit it is said to be linked once with it; while if it passes through it two or three times, as when it is encircled by two or three turns, it is said to be linked two or three times with it. We may therefore speak of the number of lines passing through a circuit multiplied by the number of turns encircling them as the total number of *linkages* of lines with the circuit, and the rate of increase or decrease of these linkages in hundreds of millions per second gives us the induced E.M.F. in volts.

For example, if a magnet had a total flux N of 2 megalines, or 2,000,000 lines, and it was inserted into or withdrawn from a coil having $n = 1000$ turns in 3 seconds, the total number of linkages would be $Nn = 2 \times 10^9$, and the average voltage

$$E = \frac{\text{number of linkages}}{\text{time in seconds} \times 10^8} = \frac{2 \times 10^9}{3 \times 10^8} = 6{\cdot}66 \text{ volts.}$$

Now suppose that our coil in Fig. 14 has no resistance, and that the core is of such a size, and wound with so many turns, that 10^8 extra lines of force are linked with it for every ampere of current which passes. Next suppose that by altering a rheostat or otherwise we cause the current to increase steadily at the rate of 1 ampere in each second. It is clear then that this means an increase of the lines linked with the coil at the rate of 10^8 per second, and this must induce an E.M.F. of one volt in the coil in such a direction as to oppose the rise of the current. There must therefore be a forward E.M.F. of one volt to overcome this back E.M.F. in order to cause the current to rise at the rate of one ampere per second.

Unit of Inductance.—*A circuit in which a change of current of* 1 *ampere causes a hundred million or* 10^8 *additional or fewer magnetic lines to be linked with it, is said to have a coefficient of self-induction or an inductance of* 1 *henry.*

Electrical Acceleration.—A circuit in which a current is increasing or diminishing in strength may be said to have an electrical acceleration. Just as a current of 1 ampere or 1 coulomb per second may be compared with a velocity of 1 ft. per second, so a change of current at the rate of 1 ampere per sec. or 1 coulomb per sec. per sec. may be compared to a mechanical acceleration of 1 ft. per sec. per sec.

We shall therefore define the electrical acceleration in a circuit as the rate of increase of current in amperes per second or coulombs per sec. per sec.

Hence the conclusion is reached that if a coil has an inductance of one henry or 10^8 lines per ampere, an electric force or E.M.F. of one volt is required to force the current to accelerate in it at the rate of one ampere per second.

Now suppose that the coil has an inductance of L henries instead of one henry, and that the current accelerates at the rate of one ampere per second. This evidently means that the number of linkages will increase at the rate of L times 10^8 per sec., and therefore L volts of back E.M.F. will be induced, which will require L volts of forward E.M.F. to produce this acceleration.

Hence we may define the inductance of a coil as the E.M.F. in volts required to force a current to flow in it with an acceleration of one ampere per second. This gives an easy and direct method of measuring the inductance of a coil, as will be seen later.

Finally, let us suppose that we have our coil of L henries inductance, and that the current is accelerated at the rate of A amperes in each second. Since L is the number of hundreds of millions of linkages produced by one ampere, and the current increases A amperes in each second, the total increase of lines of force will be LA hundreds of millions of linkages per second, and a forward E.M.F. of LA volts will be required to overcome the back E.M.F. thus produced.

We thus come to the conclusion that in order to accelerate the current in a circuit of L henries at the rate of A amperes per second, an E.M.F. V volts will be required such that

$$V = LA \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

Now look back to the law of inertia (2*a*) on p. 16, viz. $F = ma$, and compare it with the above. In words it is: The mechanical force in pounds required to change the motion of body equals mass or inertia in engineers' units multiplied by the mechanical acceleration in ft. per sec. per sec. The electrical equation says: The electrical force in volts required to change a current equals inductance in henries, multiplied by the electrical acceleration in coulombs per sec. per sec., or amperes per sec.

Is not the analogy perfect? All we have to do is to think of inductance in henries as corresponding to mass in engineers' units and call it electrical mass or inertia, and the same law, force = mass × acceleration, holds rigidly in both cases.

Inertia and Inductance Calculations.—One or two simple numerical examples will serve to show the method of working more fully, and to make the meaning of inductance quite clear.

1. A tramcar weighing 5 tons is to be started from rest and to attain a speed of 7½ miles an hour in 10 secs. What tractive force will be required?

Now, one ton = 2240 lbs. or about 70 engineers' units of mass. Hence the mass of the car m is 350 engineers' units approximately. A speed of 7½ miles per hour equals 11 ft. per sec., and if this is to be attained in 10 secs. from start, the average acceleration $a = \frac{11}{10}$ or 1·1 ft. per sec. per sec. Hence $F = ma = 350 \times 1{\cdot}1 = 385$ lbs., which is the force required to produce the acceleration, over and above that required to overcome the friction of the car.

Now suppose that we had instead of the car of 350 engineers' units of mass a coil of 350 henries inductance, and that it was desired to make the current rise to 11 amperes in 10 secs., or at the rate of 1·1 amperes per sec. Then, in exactly the same way, we should have $V = LA = 350 \times 1{\cdot}1$, or 385 volts as the E.M.F. required, over and above the voltage required to overcome the resistance of the coil.

Reverting to the tramcar, let us suppose that after it has attained its speed of 7½ miles per hour, or 11 ft. per sec., it runs into an obstacle which stops it in 2 secs. In this case the average acceleration is negative and equal to $-\frac{11}{2}$ or $-5{\cdot}5$ ft. per sec. per sec. The force required to produce this acceleration will then be $350 \times 5{\cdot}5 = 1925$ lbs., or nearly a ton, and this is the force with which it will strike the obstacle. If the stoppage is more sudden, so that it takes place in ·2 secs. instead of 2 secs., the average acceleration is increased to -55 ft. per sec. per sec., and the average force of impact to 19,250 lbs., or 8·6 tons, which is fifty times the force originally employed in getting it up to speed.

Precisely in the same way, if our coil of 350 henries has a steady current of 11 amperes in it and this current is broken slowly by a switch which stops it in 2 secs., the electrical acceleration $= -\frac{11}{2}$ or $-5{\cdot}5$ amperes per sec. The current will then try to run on with an average force of $350 \times 5{\cdot}5$, or 1925 volts, which will be the average P.D. at the coil terminals during break. If the break is made in ·2 secs. the P.D. will be 19,250 volts, and so on.

2. A shunt generator has a total magnetic flux of 2·5 megalines when a current of 4 amperes is flowing in its field coils of 2000 turns. What is the inductance of its coils? When they are switched on to 100-volt mains, what will be the rate of rise of current? and what will be the rise of pressure if the circuit is suddenly broken?

First as to the inductance. It was explained on p. 19 that the inductance of a coil in amperes is the number of hundreds of millions of linkages produced per ampere. From this it follows that the total number of linkages in this case is $2{\cdot}5 \times 10^6 \times 2000$ or 50×10^8, and as this is produced by a current of 4 amperes, the number of linkages per ampere $= \frac{50 \times 10^8}{4} = 12{\cdot}5 \times 10^8$. Hence the coil has an inductance of 12·5 henries.

If this coil is suddenly switched on to 100-volt mains we have, neglecting the resistance of the coil, $V = LA$, from which $A = \frac{V}{L}$. The current acceleration A will then be $\frac{100}{12{\cdot}5}$ or 8 amperes, and it would continue to increase at the rate of 8 amperes in each second, if it were not for the resistance of the coil. We will show later how to take this into account.

Lastly, suppose the coil is carrying a current of 4 amperes, and it is suddenly switched off by a switch which stops the current in ·01 sec. The average acceleration will then be $A = \frac{4}{{\cdot}01} = 400$ amperes per sec. Hence the average induced voltage will be $V = LA = 12{\cdot}5 \times 400 = 5000$ volts, or fifty times its normal value. This is amply sufficient to account for the breaking down of the insulation of field coils, or the flashing up of the pilot lamp on the lifting of the brushes before referred to.

Inductance of Coils with Iron Cores.—Whenever we have a coil with a closed or nearly closed magnetic circuit, we can calculate its inductance fairly approximately. If n is the number of turns in the coil, and R the reluctance of the circuit calculated from the Hopkinson formula $\mathsf{R} = \Sigma \frac{l}{A\mu}$ where l is the length of the magnetic path, A its area, and μ the permeability of the material, then the flux $N = \frac{4\pi}{10} \frac{Cn}{\mathsf{R}}$ where C is the current in amperes, and the total number of linkages $Nn = \frac{4\pi}{10\mathsf{R}} Cn^2$. Hence the inductance of the coil in henries $L = \frac{Nn}{10^8 C} = \frac{4\pi}{10^9 \mathsf{R}} n^2$. It will be seen that for a circuit of given reluctance the inductance L is proportional to the *square* of the number of turns, owing to the increase of the magnetising force, and of the number of turns encircling the flux. It must be remembered that in the case of iron the permeability varies greatly with the induction density, and hence the inductance of the coil only has a definite value when we have such an induction as to be on the straight portion of the magnetisation curve.

Of course all coils, whether with iron cores or not, have inductance, but the calculation in coils without iron is usually somewhat complex.

Measurement of Inductance.—When a coil is wound on an iron core so as to have a large inductance in comparison with its resistance, its inductance can be approximately measured in the following fairly direct manner. The coil is connected in series with two sliding rheostats and an ordinary dead-beat permanent-magnet ammeter (Fig. 16) to a battery of cells sufficient in number to give the maximum current which the coil is intended to carry. Suppose now that the resistance of

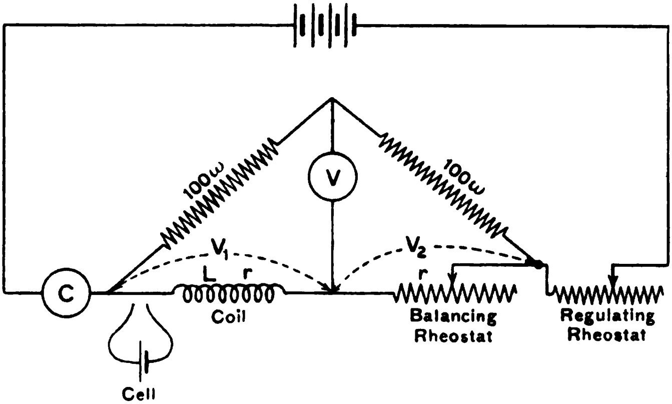

FIG. 16.—Measurement of inductance.

the coil under test is r ohms, and that by some means we have adjusted the rheostat to which it is connected (and which we will call the balancing rheostat) to have a resistance of r ohms also. Then if the second or regulating rheostat is altered so that the current C passing through the whole arrangement varies at the rate of A amperes per second, we shall have as the P.D. across the coil—

$$V_1 = rC + LA$$

and across the rheostat $V_2 = rC.$

Hence
$$V_1 - V_2 = LA$$
$$= L \text{ if } A \text{ is } 1.$$

If, therefore, we could cause the current to accelerate at the rate of one ampere per second, the difference between the voltages at the terminals of the coil and the balancing rheostat will be the inductance L in henries. In order to measure this difference the simplest device to employ would be a differential voltmeter such as is sometimes used for paralleling; but as such instruments are not generally available, one can always substitute an ordinary moving-coil voltmeter connected between the junction of the coil and balancing rheostat, and the junction of two equal coils of 100ω or 1000ω resistance, which are connected across the outer ends of the coil and rheostat, as in a Wheatstone bridge. If then a steady current is passed through the arrangement, the balancing rheostat can be adjusted until the voltmeter points to zero. On then moving the regulating rheostat so that the ammeter needle moves steadily up ampere by ampere at each tick of the clock, which can be easily done after a few trials, the voltmeter will be found to give a steady reading. Unfortunately this reading is not the induced voltage of the coil, as it would be in the case of a differential voltmeter, but it is proportional to that voltage, and the constant may readily be found by breaking the current and inserting in series with the inductive coil a single accumulator cell having an E.M.F. of about 2 volts and very low resistance. The voltmeter will then have a certain reading corresponding to 2 volts of E.M.F. in the inductive coil circuit, from which the inductive voltage can be immediately obtained. As an example one of the coils of a small transformer was connected in this way and balanced on steady currents, and on accelerating the current at the rate of 1 ampere per second the voltmeter indicated ·055 volt. On stopping the current and inserting a 2-volt secondary cell in series with the inductive coil, the voltmeter reading was

·8 volt. Hence the true voltage in the coil was $\frac{2}{\cdot 8} = 2{\cdot}5$ times the voltmeter reading, from which the inductive voltage when the current was accelerated must have been $2{\cdot}5 \times {\cdot}055 = {\cdot}138$ volt. As the acceleration was 1 ampere per second, the inductance of the coil is ·138 henry. These observations were made with an ordinary moving-coil test set.

CHAPTER III

ELASTICITY AND CAPACITY

INSTEAD of friction being the only controlling influence upon the motion of a body, we have now seen that there are at least two such influences, friction and mass or inertia; and in like manner we have seen that electric mass or inertia, which we termed inductance, has its influence upon an electric current as well as resistance. Are there any other factors to be considered?

There is one other very obvious method whereby the motion of a body may be controlled—by the attachment of a spring. Suppose, for example, that a body having no appreciable mass or friction is attached by means of a spring to a fixed point as in Fig. 17, and let it be pulled by a force of F lbs.

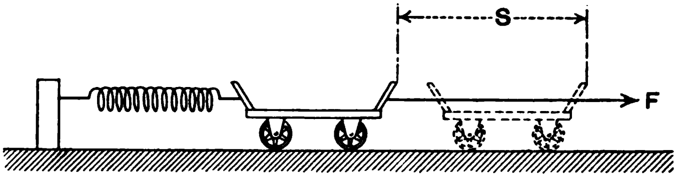

FIG. 17.—Motion of a spring.

As we know, the result will not be to produce a uniform motion as in the case of friction, nor an acceleration as in the case of inertia, but the body will simply move through a certain distance and then come to rest. If the force is increased, the movement or displacement will be greater, and in proportion to the force; but in all cases the body will come to rest, unless the force is so great as to break the spring, in which case its controlling effect will cease. A steady continuous motion is therefore impossible with a spring. Again, if the body has been displaced by a force and that force is removed, it will move back exactly to its old position if there is no friction and if the spring has no internal viscosity or 'permanent set.' We may therefore define a spring as a device which checks all steady motion, but permits a displacement proportional to the applied force.

Suppose now that we have a spring and that we pull it with a force of 1 lb. as in Fig. 18. The spring will bend or stretch through a certain distance which we will call Y feet. This displacement or yield of the spring in feet for an applied force of 1 lb. we shall call the *yield - constant* of the spring. Since the total yield or displacement of the spring is proportional to the force, the displacement in feet produced by a force of 2 lbs. will be $2Y$, or for a force F lbs., the displacement

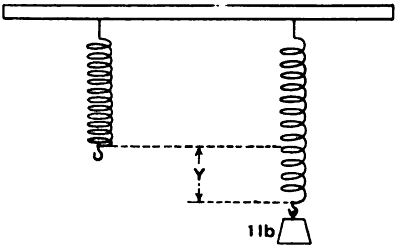

FIG. 18.—Yield-constant of spring.

$$s = YF$$

which we may call the law of the spring.

If, on the other hand, we want to know the force required to produce a certain displacement we have—

$$F = \frac{1}{Y}s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3a)$$

The properties and law of a spring can again be studied on the truck model. For this purpose the model is provided with a cross arm (Fig. 19), carrying a stretched elastic cord to which the cord representing the circuit is attached. If the truck is then used without any additional weight or friction, and a 1-lb. weight put on the scale-pan, the truck will immediately move forward a definite distance and stop, this distance giving the yield-constant on the scale of feet. In Fig. 19 the yield-constant is seen to be 0·4 ft. for 1 lb. With weights of 2, 3, &c., lbs. in the pan the yield is twice, thrice, &c., as much, while if the weight is removed the truck flies back to the zero position. On the other hand, if the weights are placed in the scale-pan on the other side, the truck is displaced by exactly equal amounts in the opposite direction.

Before inquiring whether these phenomena have any counterpart in the electrical case, we may turn for a moment to our more perfect analogy to the electrical circuit, *i.e.* the water circuit. In Fig. 20 we have the pump and pipes as before, but at the top there is shown a tank through which in the ordinary way the water could flow freely. If, however, a perfectly watertight elastic partition, such as a sheet of india-rubber, is stretched across the tank, it is obvious that the water cannot continue to flow steadily round the circuit.

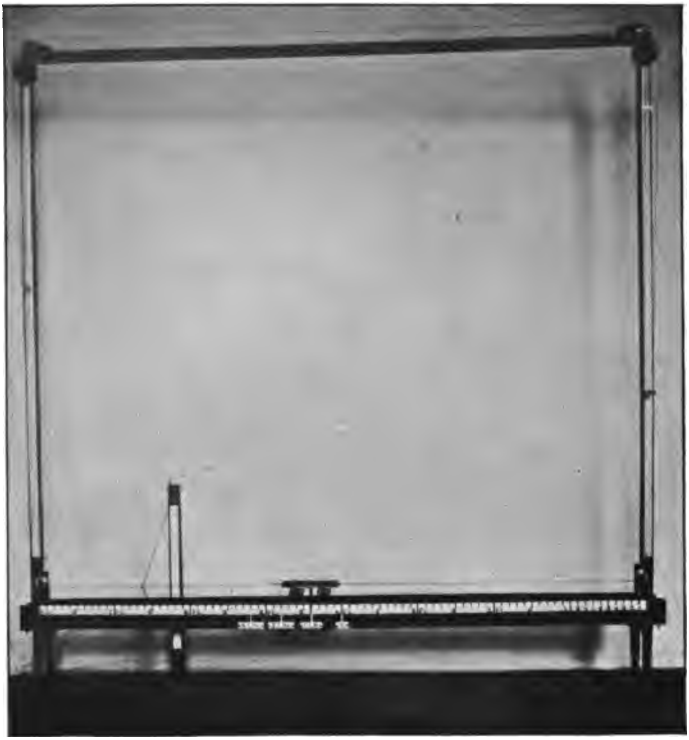

FIG. 19.—Mechanical model of electric circuit with capacity.

If the water is initially at rest, and the partition is flat, then the result of applying a pressure by the pump will be to cause a small movement or displacement of water round the circuit, and the partition will be strained to one side, as indicated by the dotted line. When this displacement has taken place, however, not the smallest steady flow of water can continue, unless the partition is leaky, or unless it breaks down under the strain. The total displacement or quantity of water circulated will evidently be proportional to the pressure or water-motive force; and if this force is removed the partition will contract and force the water backwards, so that an equal quantity flows in an opposite direction. This elastic partition therefore behaves exactly like a spring.

Now pass to the electric circuit. Our pump and pipes correspond to the battery and conductors as in Figs. 1 and 2, but what corresponds to the tank and partition? A hollow space allows water to flow and thus corresponds to a conductor, while an elastic watertight partition corresponds to an insulating sheet. Our two spaces with a watertight partition between them are thus equivalent to two metal plates with an insulating partition between them (Fig. 21), or to two sheets of tinfoil with glass or paraffined paper between them, as in a Leyden jar or condenser.

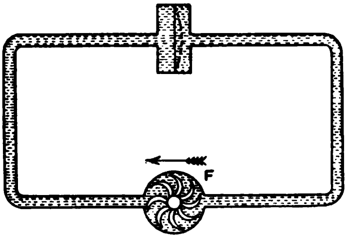

FIG. 20.—Water circuit with elastic diaphragm.

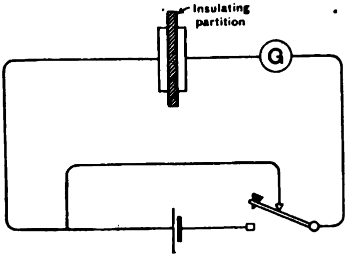

FIG. 21.—Electric circuit with condenser.

Does a condenser therefore behave as a spring? Suppose we have such a condenser joined up to a galvanometer as in Fig. 21, and apply an E.M.F. The result is a swing of the galvanometer indicating the passage of a certain quantity of electricity, after which the galvanometer returns to zero, showing that there is no steady current or flow. If twice or three times the E.M.F. is applied, the galvanometer swing is found to be twice or three times as large, *i.e.* the electric quantity or displacement is proportional to the electrical force, which exactly corresponds with the behaviour of the spring. If the E.M.F. exceeds a certain limit, it will break through the insulation and destroy the condenser, just as an excessive force broke the spring or ruptured the partition; and if, on the other hand, the force is withdrawn, a kick or discharge of equal and opposite amount to that at charge is obtained, like the recoil of the partition or spring. The analogy is thus complete.

Quantity of Electricity.—We have seen that the magnitude of a current is comparable with the speed of a moving body, and, in fact, we have called it the electric speed. If a body is moving uniformly at a speed of v feet per second, in t seconds it will have moved a total distance or displacement $s = vt$ feet.

Similarly if a current of C amperes flows steadily for t seconds, we shall have the total *quantity* or displacement of electricity $Q = Ct$. When one ampere flows for one second the quantity is called one *coulomb,* while when a body moves with a velocity of one foot per second for one second, its total displacement is one foot. Hence the electrical quantity Q in coulombs corresponds exactly to mechanical displacement s in feet.*

Law of the Condenser.—If K is the number of coulombs displaced through a condenser when one volt is applied to it, we shall have V times that displacement when an E.M.F. V volts is applied, on account of the proportionality of quantity to E.M.F.; hence the quantity in coulombs—

$$Q = KV$$

from which we find that the electric force V in volts required to produce an electric displacement or quantity Q coulombs is—

$$V = \frac{1}{K}Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

These two equations correspond exactly to those above obtained for the spring. The symbol K is called the *capacity* of the condenser in *farads,* and is the electrical displacement in coulombs produced by an applied force of one volt. It corresponds exactly, therefore, to the yield-constant Y of the spring, which was the mechanical displacement in feet produced by an applied force of one pound.

The perfect similarity between the equations representing the laws of the condenser and spring, therefore, justifies us in regarding a capacity as an electrical spring, and we shall always be able to imagine what is happening when a condenser is subjected to any varying voltage, by considering what would happen to a spring if subjected to a force varying in a similar manner.

Calculation of Capacity.—If we consider the case of an elastic partition in a water circuit, as in Fig. 20, it is evident that the quantity of water which is displaced by a given pressure is greater, the greater the area of the partition, and the thinner it is. In like manner the capacity of a condenser made up of a layer of insulation between two parallel metal plates is greater, the greater the area of the plates, and the thinner the insulation between them. Theory and experiment both show that the capacity of a parallel-plate condenser $K = k' \frac{S}{d}$ where S is the superficial area of each plate, d the distance between them, and k' a constant depending upon the material of the insulation and the units chosen. Now it may be shown, either theoretically or by experiment, that if the insulation is air, two plates each of one square foot area separated by one mil, have a capacity of ·03235 microfarad; or that two plates one square metre in area separated by one millimetre have a capacity of ·00885 microfarad. With any other material but air or ordinary gases the capacity is increased k times, and k is therefore called either the *specific inductive capacity* or the *dielectric constant* of the material used. We thus have the rules—

---

* More strictly, of course, the current in amperes or coulombs per second corresponds to the rate of flow of water in gallons or cubic feet per second instead of its speed in feet per second. In a pipe of given area, however, the rate of flow is proportional to the speed, and it is more simple to deal with motions than volumes.

$$K = \cdot 03235\, k\frac{S}{d} \text{ mfds. when } S \text{ is in sq. ft. and } d \text{ in mils}$$

$$\text{or } K = \cdot 00885\, k\frac{S}{d} \text{ mfds. when } S \text{ is in sq. metres and } d \text{ in mm.}$$

The table on p. 32 gives in the first column the value of the dielectric constant k for various substances, the second the capacity in microfarads for 1 sq. ft./mil, the third the capacity in microfarads for 1 sq. metre/mm., and the fourth the approximate dielectric strength or piercing voltage per mm.

By the help of this table the capacity of a condenser can be readily calculated, or a condenser may be designed of any given capacity. We will take one or two numerical examples.

1. A condenser is made of 50 sheets of tinfoil each 8″ × 6″ separated by two thicknesses of paraffined paper of total thickness 4 mils. Find its capacity.

The usual method of making a condenser is to pile up sheets of metal and insulation alternately like a pack of cards, and to connect alternate sheets of foil together, so that half the sheets are positive and the other half negative. It will be seen that in this way both the front and the back of the plate is used, except in the case of the two outside plates. Here the area of each plate is $\frac{8 \times 6}{144} = \cdot 333$ sq. ft., and there are 25 positive and 25 negative plates. The total area of positive or negative plate will therefore be $S = (24 \times 2 + 1) \cdot 333 = 49 \times \cdot 333$ or

TABLE I.—PROPERTIES OF INSULATING MATERIALS

Material.	Approximate Dielectric Constant *k*.	Capacity in Microfarads.		Dielectric Strength. Volts per mm.	Insulation Resistance. Megohms per cm. cube.
		Square foot mil.	Square metre mm.		
Air or gases	1	·03235	·00885	2,700	—
Paraffin wax	2·2	·0712	·0195	13,000 to 27,500	24×10^9 to 109×10^9
India-rubber, pure	2·1	·0680	·0186	—	$10\cdot9\times10^9$
„ „ vulcanised	2·7	·0874	·0239	47,600	$1\cdot5\times10^9$
Resin	2·5	·0809	·0221	13,900	—
Ebonite	2·5	·0809	·0221	53,000	—
Sulphur	3	·0971	·0265	—	—
Shellac	3	·0971	·0265	—	9×10^9
Paraffined paper	3·65	·1180	·0323	36,000	—
Gutta-percha	3·2	·1034	·0283	10,950	$\cdot025\times10^9$ to 7×10^9
Mica	6·64	·215	·0587	40,000 to 200,000	$\cdot084\times10^9$
Glass	7	·226	·0619	16,000	—
Petroleum	1·92	·0622	·0170	10,500	109×10^9
Sperm oil	3	·0971	·0265	—	—
Olive oil	3·1	·1	·0274	8,200	—
Castor oil	4·8	·155	·0425	—	—
Alcohol	26	·841	·230	—	—

16·33 sq. ft., as there are 24 intermediate plates and one outside plate for each pole. Hence we have $S = 16\cdot33$ sq. ft., $d = 4$ mils, and k' the capacity per sq. ft./mil for paraffined paper from the table $= \cdot118$ mfds., from which the capacity

$$K = k'\frac{S}{d} = \cdot118\frac{16\cdot33}{4} = \cdot482 \text{ microfarads.}$$

2. What must be the dimensions of a 1 mfd. mica condenser to stand a steady pressure of 10,000 volts ?

The dielectric strength of mica is variously given as from 40,000 to 200,000 volts per mm. Adopting a value of 100,000 volts per mm. and a factor of safety of 3, we allow 100,000 $\div$ 3 = 33,300 volts per mm., from which the thickness for 10,000 volts = ·3 mm. Now since $K = k'\frac{S}{d}$, we have $S = \frac{Kd}{k'}$. For mica $k' = \cdot0587$ microfarads per sq. metre/mm., from which $S = \frac{1 \times \cdot3}{\cdot0587} = 5\cdot1$ sq. metres. If this is made up with, say, 60 plates, we find for the area of each plate $\frac{51,000}{59} = 865$ sq.cm., or 29·4 cm. square.

Concentric Cable.—The most common example of condensers in alternate current working are the cables, which consist of two leads or conductors with insulation between. When we have a large and long cable, without any connection between the conductors or load, we have the equivalent of two metal plates of large area, with a comparatively thin layer of insulation between, and the cable therefore constitutes an elongated condenser. It is not always easy to calculate the capacity of such a cable condenser, but it is comparatively easy in the case when one conductor surrounds the other as in a concentric cable. In this case the two conductors are separated by a constant thickness of insulation, and when this thickness is small the arrangement is practically equivalent to a parallel-plate condenser rolled up into a cylinder so that one plate is outside the other. In this case, if d_1 is the diameter of the inner conductor and d_2 the diameter over the insulation, the thickness of the insulation d is $\frac{d_2 - d_1}{2}$. The mean diameter of the insulation is $\frac{d_1 + d_2}{2}$, and the mean circumference $\pi \frac{d_1 + d_2}{2}$, which is the width of the plate if it were laid out flat. Hence the total superficial area of the insulation S is circumference × length $= \frac{\pi}{2}(d_1 + d_2)l$, and the capacity $K = k'\frac{S}{d} = \pi k' \frac{d_2 + d_1}{d_2 - d_1} l$. This result is not quite correct if the inner conductor is small or the thickness of the insulation large, and the correct formula is $K = \frac{\cdot 241k}{10^6 \log \frac{d_2}{d_1}} l$ mfds.

As an illustration of this let us take the example of a concentric main 3 miles long, having an inner conductor ¾″ diameter, and a diameter over insulation of 1″; the insulation being formed of vulcanised rubber.

Here $d_1 = \cdot 75$, $d_2 = 1\cdot 00$, $l = 3 \times 5280 = 15,840$ ft. and $k' = \cdot 087$ mfds. per sq. ft./mil. The thickness of the insulation is $\frac{1 - \cdot 75}{2} = \cdot 125''$ or 125 mils, the mean diameter

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$\frac{1 + {\cdot}75}{2} = {\cdot}875''$, and the mean circumference $= \pi \times {\cdot}875 = 2{\cdot}75'' = {\cdot}229$ ft. The cable is therefore approximately equivalent to two plates each having a superficial area of $15{,}840 \times {\cdot}229 = 3620$ sq. ft., and separated by 125 mils of vulcanised rubber, and $K = k' \frac{S}{d} = {\cdot}087 \frac{3620}{125} = 2{\cdot}52$ microfarads.

By the more exact formula we have $l = 15{,}840 \times 30{\cdot}48 = 4{\cdot}83 \times 10^5$ cm. and the capacity—

$$K = \frac{{\cdot}241k}{10^6 \log \frac{d_2}{d_1}} l = \frac{{\cdot}241 \times 2{\cdot}7}{10^6 \times {\cdot}1250} \times 4{\cdot}83 \times 10^5$$

$$= 2{\cdot}52 \text{ microfarads}$$

If such a cable is charged with 10,000 volts a quantity $Q = KV = 2{\cdot}5 \times 10^{-6} \times 10{,}000$ or ·025 coulomb passes in.

Measurement of Capacity.—From the fundamental equation of the condenser $V = \frac{1}{K} Q$, we have $K = \frac{Q}{V}$, or the capacity in farads is equal to the electrical displacement in coulombs divided by the electrical force in volts; just as the yield-constant of a spring is the displacement in feet divided by the force in lbs. which produces it. If $V = 1$, $K = Q$, or K is the number of coulombs which passes through a condenser for an applied force of one volt, as defined on p. 30. Hence, if we could measure this quantity directly, we could at once find the capacity of a condenser, but the measurement of electric quantity is not very easy.

Suppose, however, that we make the P.D. across the condenser terminals increase steadily at the rate of one volt per second. For each volt K coulombs will pass through the condenser, and hence the quantity will pass at the rate of K coulombs per second, or K amperes. The result, therefore, of applying a P.D. increasing at the rate of one volt per second to a condenser is to produce a steady current in it, this current in amperes being equal to the capacity in farads. The current in micro-amperes will therefore be equal to the capacity in microfarads. Now there are plenty of galvanometers, even of the pivoted type, nowadays which are or can be calibrated directly in micro-amperes. If such a galvanometer, joined in series with a condenser and in parallel with a voltmeter, as in Fig. 22, is connected between one end of a slide wire and the contact which slides along the wire, then when a steady current flows through the wire, and the sliding contact is moved steadily along it, the P.D. will rise steadily, and it is easy to move the slides so that the voltmeter needle goes up volt by volt at each tick of a clock. Under these circumstances the capacity of the condenser in microfarads is given directly by the galvanometer reading in micro-amperes.

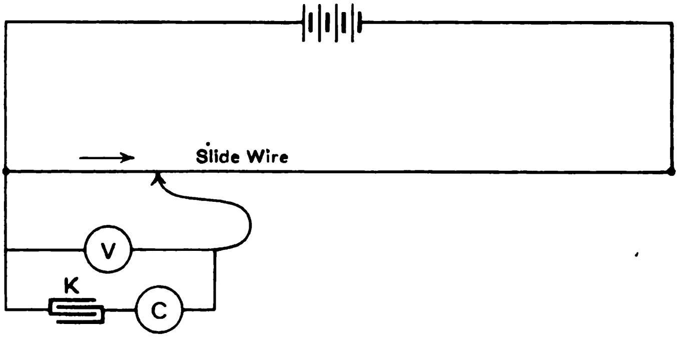

FIG. 22.—Measurement of capacity.

The method as described here is only capable of an accuracy of the order of five per cent., but it can be modified in such a way as to give accurate results. Our object here, however, is not to describe testing methods of high accuracy so much as to give those which depend on and illustrate fundamental principles.

CHAPTER IV

CONSEQUENCES OF THE ELECTRICAL AND MECHANICAL COMPARISONS. ENERGY AND POWER

WE have now completed our comparison of electrical and mechanical quantities with one exception, viz. mutual induction, which it will be convenient to reserve till later. As it is of the greatest possible importance that these comparisons should be thoroughly assimilated, a summary sheet (see pp. 38, 39), which has been employed in the author's classes for some years past, is here included, and illustrated by the accompanying Plate I.

The comparisons which we have developed between the mechanical and electrical quantities are put briefly in parallel columns, and they should be read paragraph by paragraph on the two sides alternately. There follows a summary of the laws of the electric circuit, of a moving body, and of a rotating body in tabular form. The third column dealing with rotating bodies need not be taken any notice of, but is included for the sake of symmetry, and for those who have gone a little further into the study of mechanics.

The bulk of the table requires no further explanation, but the items under the headings 'Energy' and 'Power,' and other comparisons, may be briefly referred to.

Energy or Work.—How do we measure energy or work in mechanics? By definition work is done when a body is displaced under the action of a force which is in the direction of the displacement. An amount of work called one foot-pound is done by a force of 1 lb. acting through a displacement of 1 ft. Or—

$$e = Fs \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

where e is the energy or work in ft.-lbs., F is the force in lbs.,

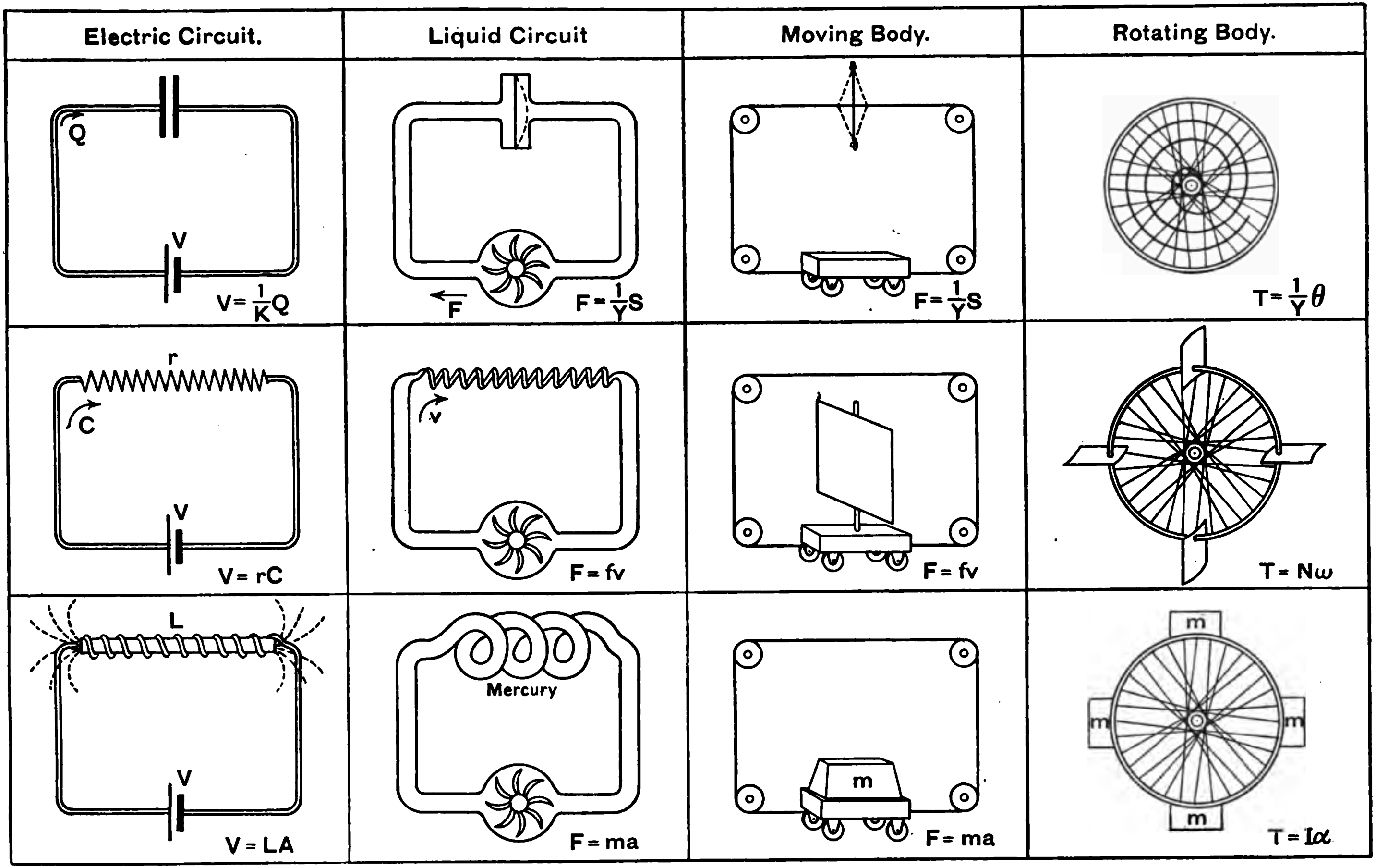
Electric Circuit.
Liquid Circuit
Moving Body.
Rotating Body.
Q
V
$V=\frac{1}{K}Q$
F
$F=\frac{1}{Y}S$
$F=\frac{1}{Y}S$
$T=\frac{1}{Y}\theta$
r
C
V
$V=rC$
v
$F=fv$
$F=fv$
$T=N\omega$
L
V
$V=LA$
Mercury
$F=ma$
m
$F=ma$
m
m
m
m
$T=I\alpha$

PLATE I.

TABLE II

COMPARISON OF MECHANICAL AND ELECTRICAL QUANTITIES

MECHANICAL	ELECTRICAL
FORCE tends to produce or alter motion of matter. Symbol F. Unit of force, 1 **lb.**	**E.M.F.** or **P.D.** tends to produce or alter current. Symbol V. Unit of E.M.F. or P.D. 1 **volt.**
DISPLACEMENT. — Total movement of matter. Symbol s. Unit 1 **foot.**	**QUANTITY.** — Total flow of electricity Symbol Q. Unit 1 **coulomb.**
SPEED.—Rate of motion of matter. Symbol v. Unit speed 1 **ft. per sec.**	**CURRENT.** — Rate of flow of electricity. Symbol C. Unit 1 **ampere** or 1 coulomb per sec.
ACCELERATION.—Rate of increase of speed. Symbol a. Unit 1 **ft. per sec. per sec.**	**ELECTRICAL ACCELERATION.**—Rate of increase of current. Symbol A. Unit 1 **ampere per sec.** or 1 **coulomb per sec. per sec.**
FRICTION always resists motion. If motion takes place against friction, mechanical energy is expended and converted into heat.	**RESISTANCE** always opposes flow of current. If current flows through resistance, electrical energy is expended and converted into heat.
With solid friction the force required to overcome it when motion is once started is independent of the speed; **with fluid friction at low speeds, or magnetic friction, force is proportional to speed**; and with fluid friction at high speeds it is proportional to the square of the speed.	With metallic conductors at a fixed temperature the voltage required to overcome the resistance is proportional to the current.
For the second or average case— The '**friction constant**' of a body is the force in lbs. required to maintain it at a steady speed of 1 ft. per sec. Symbol f. Then force required for any speed, $F = fv$ (mechanical 'Ohm's law')	The **resistance** of a conductor in ohms is the P.D. in volts required to maintain a steady current of 1 ampere in it. Symbol r. Then P.D. required for any current, $V = rC$ (Ohm's law)
MASS OR INERTIA.—Tends to oppose any *change* of velocity. A stationary mass resists starting, and when any speed has been attained resists its stoppage. If moving mass is suddenly checked by obstacle, it may break through it.	**INDUCTANCE OR SELF-INDUCTION** tends to oppose any *change* of current. An inductive coil resists starting of current, and when any current is flowing resists its stoppage. If current is suddenly checked in an inductive circuit, it may break through gap or insulation.
Engineers' unit of mass 32·2 lbs. A body has one engineers' unit of mass when a force of 1 lb. gives it an acceleration of 1 ft. per sec.	Unit of inductance 1 henry. A circuit has an inductance of 1 henry when on increasing the current in it by 1 ampere the flux linked with the circuit increases by 10^8 maxwells or C.G.S. lines. If the current increases at the rate of 1 ampere per sec., the flux linked with the circuit is increased at the rate of 10^8 lines per second, producing a back E.M.F. of 1 volt.
Symbol for mass, m, Engineers' units. Hence the force F in pounds required to produce an acceleration of a ft. per sec. per sec. in a body of mass m units is $F = ma$	Symbol for inductance, L henries. Hence the voltage required to produce an electric acceleration of A amperes per sec. in a circuit of inductance of L henries is $V = LA$
ELASTICITY resists the movement of a body in either direction with a force proportional to the displacement. No steady motion can exist with a perfect spring. If the applied force exceeds a certain limit, the spring will break and no longer resist the motion.	**CAPACITY** resists the flow of electricity in either direction with a voltage proportional to the quantity. No steady current can flow through a perfect condenser. If the applied voltage exceeds a certain limit, the insulation of the condenser will be broken through and will no longer prevent the passage of current.
Unit spring permits a displacement of 1 ft. for force of 1 lb.	Unit of capacity 1 farad permits a quantity of 1 coulomb to pass for E.M.F. of 1 volt.
The **yield constant** of a spring is the displacement in feet for an applied force of 1 lb. Symbol Y.	The **capacity** of a condenser is the quantity in coulombs for an applied E.M.F. of 1 volt. Symbol K.
Hence if a spring having a yield constant Y ft. per lb. is acted on by a force F lbs., the displacement in feet will be $s = YF$, or force required for displacement s will be $F = \frac{1}{Y}s$	Hence if a condenser having a capacity K farads is acted on by an E.M.F. of V volts the quantity in coulombs will be $Q = KV$ or E.M.F. required for quantity Q will be $V = \frac{1}{K}Q$

TABLE II—*continued*

COMPARISON OF ELECTRICAL AND MECHANICAL RELATIONS

	ELECTRIC CIRCUIT.	MOVING BODY.	ROTATING BODY.
Symbols and Units.	Electromotive Force V volts Electric Quantity Q coulombs (ampere seconds) Electric Current C amperes (coulombs per sec.) Electric Acceleration A amperes per second (or coulombs per sec. per sec.) Resistance r ohms Inductance L henries (1 henry = 10^8 lines linked per ampere) Capacity K farads Energy J joules Power or Activity w watts (joules per sec.)	Force F lbs. Displacement s feet Speed or Velocity v ft. per sec. Acceleration a ft. per sec. per sec. Friction Constant f lbs. for 1 ft. per sec.‖ Mass or Inertia m engineers' units of 32·2 lbs. Yield Constant Y feet per lb. Energy e ft.-lbs. Power or Activity P ft.-lbs. per sec.	Torque or moment T lb.-feet Angular displacement θ radians Angular velocity ω radians per sec. Angular acceleration α radians per sec. per sec. Friction moment N lb.-ft. for 1 radian per sec. Moment of inertia I units* Torsion Constant y radians per lb.-ft. Energy e ft.-lbs. Power or Activity P ft.-lbs. per sec.
Fundamental Equations.	Condenser $V_1 = \frac{1}{K} Q$ or $Q = KV_1$ Resistance $V_2 = rC$, or $C = \frac{V_2}{r}$ Inductance $V_3 = LA$, or $A = \frac{V_3}{L}$	Spring $F_1 = \frac{1}{Y} s$ or $s = YF_1$ †Friction $F_2 = fv$, or $v = \frac{F_2}{f}$ Inertia $F_3 = ma$, or $a = \frac{F_3}{m}$	Torsion $T_1 = \frac{1}{y}\theta$ or $\theta = yT_1$ †Friction $T_2 = N\omega$, or $\omega = \frac{T_2}{N}$ Inertia $T_3 = Y\alpha$, or $\alpha = \frac{T_3}{Y}$
Activity or Power.	§In all cases $w = VC$ for resistance $w = \frac{V^2}{r} = C^2 r$	In all cases $P = Fv$ for friction $P = \frac{F^2}{f} = v^2 f$	In all cases $P = T\omega$ for friction $P = \frac{T^2}{N} = \omega^2 N$
Other Analogies.	Total flux C.G.S. lines or maxwells linked with circuit $N = LC$ Energy stored in condenser $J = \frac{1}{2}KV^2$ Energy stored in inductance $J = \frac{1}{2}LC^2$	Momentum $= mv$ Energy stored in spring $e = \frac{1}{2}YF^2$ Kinetic energy in moving mass $e = \frac{1}{2}mv^2$	Moment of Momentum $= I\omega$ Energy stored in spring $e = \frac{1}{2}yT^2$ Kinetic energy of rotating body $e = \frac{1}{2}I\omega^2$

‖ In the case of solid friction, force is independent of speed; with fluid friction at low speeds or magnetic damping, force is proportional to speed; and with fluid friction at high speeds force is proportional to square of speed. The second case is here assumed.

* The moment of inertia $I = \Sigma mr^2$ where m is the mass in engineers' units and r the radius in feet.

† Under the assumption that frictional force or torque is proportional to speed.

§ Note that the power is here not the *average* but the *instantaneous* rate of flow of energy, if the current varies.

and s the displacement in feet. Now according to our analogies a mechanical force of 1 lb. corresponds to an electrical force of 1 volt, and a displacement of 1 ft. to a quantity of 1 coulomb. If this comparison holds, therefore, the displacement of a certain quantity of electricity by an electric force should imply a certain amount of energy, as we know it does. The work done when 1 volt causes a displacement of 1 coulomb is termed 1 joule, and we then have—

$$J = VQ \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

where J is the energy in joules, V the force in volts, and Q the quantity in coulombs. This again corresponds exactly to the mechanical equation.

Or, in words, we have—mechanical work in ft.-lbs. equals mechanical force in pounds multiplied by the mechanical displacement in feet; and electrical work in joules equals electrical force in volts multiplied by electrical displacement in coulombs.

Power.—This is defined as the *rate* of doing work. If a body moves steadily with uniform velocity, under the action of a steady force F lbs., then the work done is $e = Fs$ ft.-lbs. Hence the power—

$$P = \frac{e}{t} = \frac{Fs}{t} = F \times \frac{s}{t} = Fv \quad . \quad . \quad . \quad . \quad (5a)$$

or the power in ft.-lbs. per sec. is the product of the force in lbs. and the velocity in ft. per sec.

But force F in lbs. corresponds to V volts, and v in ft. per sec. to C amperes. Hence the power in electrical units (joules per second or *watts*) should be—

$$w = VC \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

which is familiar to us all, and may be expressed by saying that the electrical power in joules per second or watts is equal to the product of the electrical force in volts and the electrical speed in coulombs per second or amperes.

If the body moves against a friction-constant f we know that $F = fv$, and hence—

$$P = Fv = fv \times v = fv^2 \quad . \quad . \quad . \quad (6a)$$

Writing w for P, $V = rC$ for F, and r for f, we have—

$$w = rC^2 \text{ or } C^2r \quad . \quad . \quad . \quad . \quad (6b)$$

as is well known. Hence in the case of mechanical friction the power in ft.-lbs. per sec. is the product of the friction-constant and the square of the velocity in ft. per sec.; while in the case of electrical friction or resistance, the power in joules per sec. or watts is the product of the electrical friction-constant or resistance in ohms, and the square of the electrical velocity or current in amperes.

Energy of Motion.—Next suppose a mass m is acted on by a steady force F, producing an acceleration $a = \frac{F}{m}$. In a time t seconds the body will have attained a velocity $v = at$ ft. per sec. But the *average* velocity during this time will only be half of this, as it is zero at the commencement and v at the end of the time. Hence the distance travelled in the t secs. = average velocity × time or $s = \frac{1}{2}v \times t = \frac{1}{2}at^2$, and the work done by the force is

$$\begin{aligned} Fs &= F \times \tfrac{1}{2}v \times t \\ &= ma \times \tfrac{1}{2}v \times t = \tfrac{1}{2}mv \times at = \tfrac{1}{2}mv^2 \end{aligned}$$

This is the work done by the force in moving the body, and it is stored up in the body as energy of motion or *kinetic* energy. Hence the kinetic energy of a moving body in ft.-lbs. is—

$$e = \tfrac{1}{2}mv^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

or half the product of its mass in engineers' units and the square of its velocity in feet per second.

Energy of Inductance.—If we translate the equation just obtained into electrical language we write J joules for e ft.-lbs., L henries for m engineers' units of mass, and C amperes for v ft. per sec., and we have—

$$J = \tfrac{1}{2}LC^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

which is the energy that has to be expended in forcing the current to accelerate until it reaches the electrical velocity C; and this energy may be called the electro-kinetic energy of the circuit.*

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* Strictly speaking, this energy is not stored in the circuit, but principally in the medium surrounding it. We need not enter into this here, but it will be dealt with later (see Chapter XVII), and for the moment it will be sufficient to consider it as being associated with the coil, as kinetic energy is with the mass.

Finally, again, if we have a spring and stretch it by applying a force, this force $F = \frac{1}{Y}s$. The average force during the stretching process is, however, only half this, or $\frac{1}{2Y}s$, since the resisting force of the spring is zero at first and increases proportionally to the amount of stretching. The total energy given to the spring is therefore the average force × distance

$$= \tfrac{1}{2}Fs = \frac{s^2}{2Y} = \tfrac{1}{2}YF^2 \quad . \quad . \quad . \quad . \quad (8a)$$

In like manner, for a condenser K farads corresponds to Y, and so on, therefore we may write—

$$J = \tfrac{1}{2}VQ = \frac{Q^2}{2K} = \tfrac{1}{2}KV^2 \quad . \quad . \quad . \quad (8b)$$

Each of these electrical equations could have been obtained by reasoning directly from the fundamental electrical relations, and these examples are quite sufficient to show how perfectly the analogies hold, and how they simplify the realisation of electrical problems. So long as we bear in mind the limitations *re* friction, anything which we deduce from the three mechanical laws and the law of continuity *must* have its counterpart among those electrical phenomena which depend on the three fundamental laws and Kirchhoff's law, and any experimental phenomena which we notice with springs, friction, and weights must have electrical phenomena corresponding to them.

It will be well to note, before leaving this part of the subject, that wherever energy or power is concerned, it depends upon the square of the displacement, velocity, &c., or in the electrical case, the square of the P.D., current, or quantity. The importance of this will be seen later. Meanwhile it may be pointed out that this result is to be expected from a very simple consideration. The energy or power required to accelerate a mass, to overcome friction or bend a spring, or to pass a current through a resistance or charge a condenser, is evidently independent of the direction of the motion.

But if this is the case it must depend upon the square or fourth or sixth power of the motion, for if it were proportional to an odd power it would become negative when the motion reversed. This was first realised by Joule when conducting his famous researches on the heating effects of an electric current.

CHAPTER V

COMBINATIONS OF FRICTION, INERTIA, AND ELASTICITY, OR OF RESISTANCE, INDUCTANCE, AND CAPACITY

UP to the present we have always been dealing with ideal rather than with real conditions. We have imagined bodies having friction only, inertia only, or elasticity only; or circuits having only one of the three attributes—resistance, inductance, and capacity. But this is obviously seldom, if ever, true. A body without both mass and friction is unknown, and a circuit without resistance and a certain amount of inductance is also unknown. It is therefore necessary for us to deal with combinations of these quantities, and this will be found fairly easy and interesting.

The most usual and simple combination is perhaps that of a body having both inertia and friction. As an illustration of a body which has approximately the kind of friction we are considering, we may take a screw-propelled boat, in which a definite rate of revolution of the screw corresponds to a more or less definite propulsive force, and find how the boat will get up speed when the screw is started. It is perfectly easy to measure the friction-constant of such a boat by towing it along at a speed of 1 ft. per sec. with a spring balance, the reading of this spring balance in pounds being the friction-constant f directly. The mass of the boat m can also be obtained by weighing it and dividing the result in pounds by 32·2 to reduce it to engineers' units.

Now when the boat is moving the force required to overcome the friction in pounds is $F_2 = fv$, and the force required to accelerate it at a rate a ft. per sec. is $F_3 = ma$. Consequently the total force required to keep it moving and also to accelerate it is $F = F_2 + F_3 = fv + ma$. Hence if the boat is stationary and a definite propulsive force F is applied, it will move in such a manner that the sum of fv and ma is always equal to F.

Consider, first, for a moment what would happen if the boat possessed friction only without inertia. Then we should have $F = F_2 = fv$ only, from which $v = \frac{F}{f}$, or the boat would start off with a constant velocity $\frac{F}{f}$ the moment that the force was applied. This is represented by the horizontal line in the diagram, Fig. 23, in which time is plotted horizontally and velocity vertically. Next, suppose that the boat had inertia only without friction. Then $F = F_3 = ma$ only, from which $a = \frac{F_3}{m}$, or the boat would start and increase its speed with a constant acceleration $a = \frac{F_3}{m}$. This means that it would have a speed of $\frac{F_3}{m}$ at the end of the first second, of $\frac{2F_3}{m}$ at the end of the next second, of $\frac{3F_3}{m}$ at the end of the third, and so on *ad infinitum*. In other words, the speed would go on steadily increasing, as shown by the sloping line, without any limit.

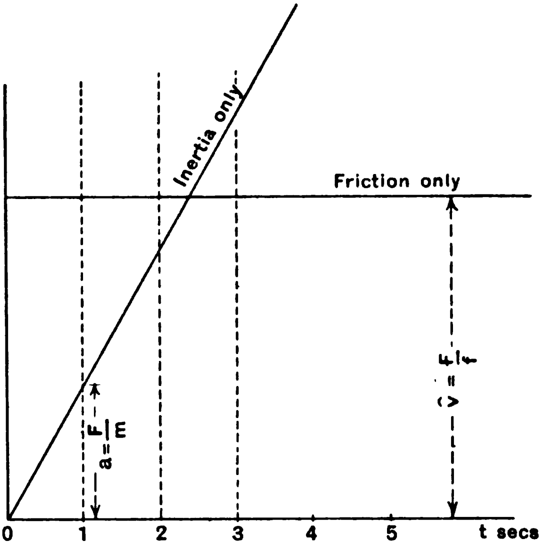

FIG. 23.—Motion of boat with friction only or with inertia only.

Now consider the actual case of the friction and inertia together, for which $F = F_2 + F_3 = fv + ma$. We can write this $F_3 = F - F_2$ or $ma = F - fv$, from which $a = \frac{F - fv}{m}$.

At the moment of first applying the force the boat has not begun to move, and $v = 0$. Hence $F_2 = fv$ is zero, or there is no force exerted in maintaining the speed of the boat against friction, and the whole of the force F is available for accelerating the boat. Therefore $a = \frac{F}{m}$, as when we supposed friction to be absent; the boat will start as if no friction existed, and the velocity curve will follow the line corresponding to no friction. As soon, however, as we have any appreciable speed v, a force $F_2 = fv$ is required to maintain it, and this must be deducted from the propulsive force F in order to get the part F_3 left for acceleration purposes.

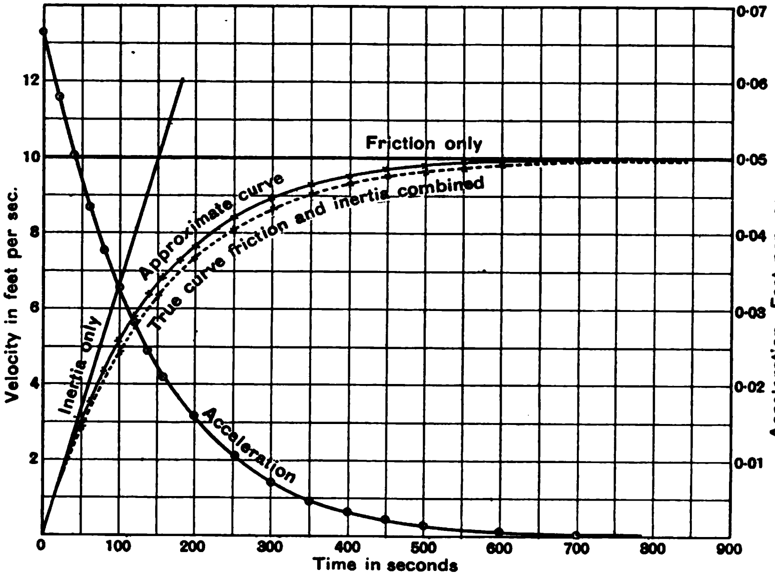

FIG. 24.—Motion of boat with both friction and inertia.
$F = 20$; $f = 2$; $m = 300$.

At this point it will be well to take a numerical example in order to see how the complete velocity curve can be obtained. We will take the case of a motor-boat weighing 9660 lbs., and having engines which produce a propelling force of 20 lbs. We will suppose that the friction-constant is 2 lbs. for a velocity of 1 ft. per sec.

The mass of the boat will be $\frac{9660}{32{\cdot}2} = 300$ engineers' units of mass. Then at starting we have $v = 0$, and therefore $F_2 = 0$, hence $a = \frac{F}{m} = \frac{20}{300} = {\cdot}0667$ ft. per sec. per sec. In 20 secs. after starting the velocity at this rate of acceleration would be $20 \times {\cdot}0667 = 1{\cdot}333$ ft. per sec., and this would require the application of a force $F_2 = fv = 2 \times 1{\cdot}333 = 2{\cdot}667$ lbs., leaving $20 - 2{\cdot}667$ or $17{\cdot}333$ lbs. for acceleration. The remainder of the calculation will be obvious from the annexed table. It must, of course, be remembered that the method is not rigidly correct, as even during the first twenty seconds the acceleration begins to diminish, and consequently the velocity will not be quite so great at the end of that time as we have calculated. By taking ten-second or five-second intervals we should, of course, get a more accurate result, but it would

TABLE III.—MOTOR-BOAT. $F = 20, f = 2, m = 300.$

Time from start in Seconds $= t$.	Speed in ft. per Sec. $= v$.	Force to overcome Friction $F_2 = fv$.	Force available for Acceleration $F_3 = F - F_2$.	Acceleration $= \frac{F_3}{m}$.
0	0	0	20	= ·0667
20	1·333	2·666	17·333	·058
40	2·45	4·9	15·1	·0503
60	3·51	7·02	12·98	·0433
80	4·38	8·95	11·25	·0375
100	5·13	10·26	9·74	·0325
120	5·78	11·56	8·44	·0281
140	6·34	12·68	7·32	·0244
160	6·83	13·86	6·14	·0205
180	7·24	14·48	5·52	·0184
200	7·61	15·22	4·78	·0159
250	8·4	16·8	3·2	·0106
300	8·93	17·86	2·14	·0071
350	9·29	18·58	1·42	·0047
400	9·53	19·06	0·94	·0031
450	9·69	19·38	0·62	·0021
500	9·79	19·58	0·42	·0014
600	9·93	19·86	0·14	·00047
700	9·977	19·95	0·05	·00017
800	9·994	&c.	&c.	&c.

not differ greatly from the foregoing. In Fig. 24 the results given in the table have been plotted, together with the correct curve obtained by more accurate calculation; these show that the velocity of the boat increases rapidly at first along the line corresponding to inertia only, and afterwards more slowly, approaching more and more nearly to (but never quite reaching) the value $\hat{v} = \frac{F}{f} = 10$ ft. per sec., which it would have attained immediately had the boat been devoid of inertia.* In other words, when the inertia and friction are combined the velocity curve coincides at first with the inertia line, but soon bends over and approaches the friction line.

One other point may be noticed about the curves. If we take the two velocity curves for inertia or friction only, they cross one another in our case at a point 150 seconds from the start. This is the time which it takes for the velocity of the frictionless mass to become accelerated to the same value as it would attain if it were endowed with friction without inertia. This time can be readily calculated, for if we denote it by T seconds, we know that the velocity attained under acceleration is $v = aT$, and since $F = ma$, $a = \frac{F}{m}$, and $v = \frac{F}{m}T$. But with friction alone $v = \frac{F}{f}$, hence for these two velocities to be equal $\frac{F}{m}T = \frac{F}{f}$, from which $T = \frac{m}{f}$ seconds. In our case $m = 300$ and $f = 2$, whence $T = 150$ seconds, as shown by the diagram. This time, given by dividing the mass by the friction-constant, may be called the *time-constant* of the combination of mass and friction.† Of course, when the velocity is increasing under the combined influence of inertia and friction it does not reach the full velocity in this time, but it does reach a definite fraction of it. In our diagram we find that the velocity at the end of the 150 secs. is 6·5 ft. per sec., or ·65 of its final value of 10 ft. per sec., but this is slightly too high owing to our having used such long intervals in our calculation, and the correct fraction should be ·634, as may be shown later. Whatever be the force, inertia or friction, it will be found that in the time $T = \frac{m}{f}$ the velocity always attains a fraction ·634 or 63·4% of its final value $\vartheta = \frac{F}{f}$, and thus the time T gives us an idea of the rate of rise.

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* The symbol $\hat{v}$ is used to denote the final or *maximum* value of the velocity v, and the sign ^ above a letter will always be employed for its maximum value, representing as it does the top or crest of a wave or oscillating motion.

† As a matter of fact the term 'time-constant' has only been employed for the electric circuit having resistance and inductance, but its application here is obviously justifiable.

Before turning to the electrical problem we may show how the curve may be obtained very rapidly by the use of drawing instruments after a few simple preliminary calculations. Our equation was $F = F_2 + F_3 = fv + ma$, from which we get $ma = F - fv$ and $a = \frac{F - fv}{m}$. If we now divide the numerator and denominator of this fraction by f its value will remain unchanged, and we shall have $a = \frac{\frac{F}{f} - v}{\frac{m}{f}}$.

But $\frac{F}{f}$ is the velocity which the body would have with friction only, *i.e.* the ultimate or maximum value of the velocity to which it will attain under the actual conditions. If we denote this maximum velocity $\frac{F}{f}$ by ϑ, and at the same time remember that the denominator $\frac{m}{f}$ is what we have just called the time-constant T, our expression becomes—

$$a = \frac{\vartheta - v}{T} \quad \text{where} \quad \vartheta = \frac{F}{f} \quad \text{and} \quad T = \frac{m}{f} \qquad (9a)$$

The physical meaning of this equation is that the body is accelerated at each instant by an amount equal to the difference between its frictional or final velocity and its actual velocity, divided by its time-constant.

Now suppose in Fig. 25 that we make a triangle by marking off a line T units long horizontally to represent the time-constant T secs., and another vertically $\vartheta - v$ units long. Then if we complete the triangle, since its height is $\vartheta - v$ for a base T, the height of the sloping line will be $\frac{\vartheta - v}{T}$ for a distance 1 sec. horizontally, and this is the acceleration *a*. The height of the line for any time *t* secs. is then evidently $\frac{\hat{v} - v}{T}t$ or *at*. But this is precisely the amount by which the velocity would increase in the time *t*, since *a* is the increase of velocity in each unit of time, and the sloping line therefore represents the velocity curve for a short distance until the acceleration is altered by the new value of *v*. Next take a piece of squared paper as in Fig. 26, in which the horizontal axis represents time and the vertical axis velocity, and draw a horizontal line *PQ* at a height $\hat{v} \doteq \frac{F}{f}$ from the base representing the final velocity of the body. Along this line mark off a distance *Pa* representing the time-constant $T = \frac{m}{f}$. Then if a straight line is drawn from the origin *O* to the point *a*, this line will coincide with the beginning of the velocity curve, for it is the third side of a triangle of which the vertical side is $\hat{v} - v$ (*v* being zero as the body is supposed to start from rest) and the horizontal side *T*. The velocity will therefore increase at first along the line *Oa*, but we must only take a short length of it, *OA*, since the acceleration will change as soon as the body has got up an appreciable speed. But now, if a vertical line is carried up from *A* to meet the line *PQ* at *x*, then *Ax* is the difference between the final velocity $\hat{v}$ and the velocity actually attained *v*, or is equal to $\hat{v} - v$. Consequently, if we mark forward the same distance *T* from *x* to *b*, and draw a straight line from *A* to *b*, we have a triangle in which the vertical side is $\hat{v} - v$ and the horizontal side *T*, and the velocity will continue to rise along the line *Ab*. We therefore draw a portion of this line *AB* and mark off a point *c* on *PQ*, which is a distance *T* in front of *B*. The next segment of the curve will be along *BC*, and so on. We thus get a curve which rises rapidly at first and bends over more and more towards the horizontal as it gets

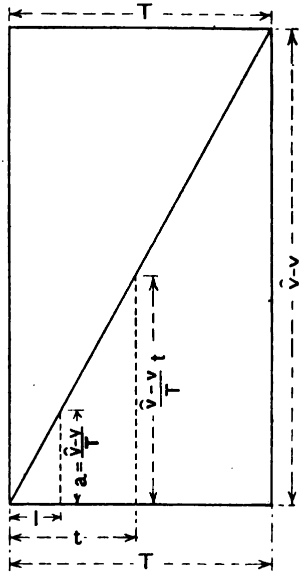

FIG. 25. — Acceleration with friction and inertia.

nearer and nearer to *PQ*, so that it continually approaches this line without ever actually meeting it.* This curve should be compared with Fig. 24, and it will be seen that it is such that at every instant it is directed to a point a distance *T* in front of it on the line *PQ*. In fact, if *PQ* represents one side of a road and a person walks along it, then a second person starting from a point *O* on the opposite side of the road, and always walking straight towards him, will move over a curve identical in form

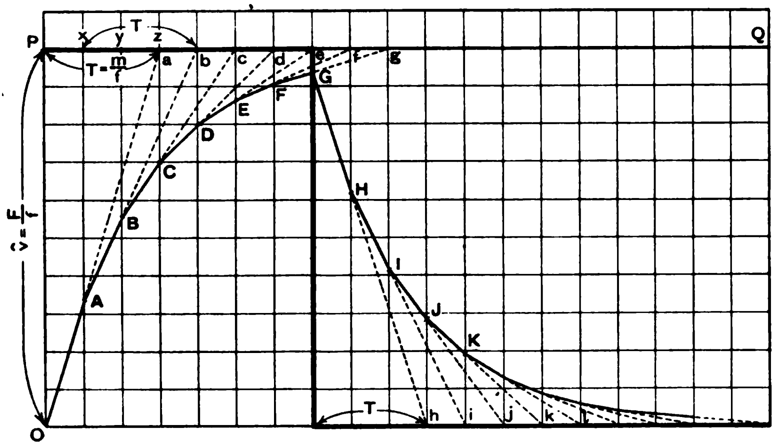

FIG. 26.—Graphical construction for velocity curve.

with the velocity curve, providing that the first person always keeps ahead of the second by a distance *T* measured along the direction of the road.

If the velocity has risen to a certain value and the propelling force is suddenly removed, as by the sudden stoppage of the engines of the boat in our former example, the ultimate velocity will, of course, be $\hat{v} = \frac{F}{f} = 0$. The horizontal line through *O* is therefore the final velocity line, and we can see how the speed will fall by joining the point *G* at which the force was removed to a point *h* on the final velocity line, which is a distance *T* in front of *G*, and repeating the process. It is, in fact, precisely as if when the second person had followed the first for some distance, the latter had suddenly crossed to the other side of the road, always keeping the same distance ahead. Consequently, if the force is suddenly applied at *O* and suddenly withdrawn at *G*, the velocity, instead of rising instantly to *P* and falling suddenly on the cessation of the force, rises slowly and falls slowly, as shown by the curve. This agrees very well with what is observed in practice, and with what we saw in Figs. 8*a* and 8*b*.*

---

* This method of obtaining the velocity curve is a modification of that devised by Dr. Sumpner some years ago for an inductive coil.

Rise of Current in an Inductive Coil.—If the foregoing mode of obtaining the variation of velocity of the boat has been understood the case of an inductive coil will be perfectly simple. For if we have a coil with a resistance of *r* ohms and an inductance of *L* henries, and apply a voltage *V* to its terminals, this voltage will be employed in forcing the current through the resistance *r* and in accelerating it against the inductance *L*. Hence our equation becomes $V = V_2 + V_3 = rC + LA$, which resembles exactly the equation $F = F_2 + F_3 = fv + ma$, which we had for the moving body. The electrical problem can therefore be solved in exactly the same way as that of the boat; in fact, if we were to apply a P.D. of 20 volts to a coil having a resistance of 2 ohms and an inductance of 300 henries, the value of the current at every instant would be given by the velocity in feet per second at the corresponding instant in Fig. 24. Let us next take the case of the shunt generator referred to on p. 22 which had a flux of 2·5 megalines per pole for an exciting current of 4 amperes, in a coil of 2000 turns, at 100 volts. This gave us an inductance *L* of 12·5 henries and a resistance *r* of $\frac{100}{4} = 25$ ohms, from which the time-constant $T = \frac{L}{r} = \frac{12{\cdot}5}{25} = {\cdot}5$ sec. With resistance only the current $C = \frac{V}{r} = 4$ amperes, which would be steady from the moment of switching on, while with inductance only we have $V = LA$, from which $A = \frac{V}{L} = \frac{100}{12{\cdot}5} = 8$ amperes per sec., as we saw on p. 22, and the current would go on increasing at this rate indefinitely if there were no resistance. As the current grows in the second case at 8 amperes per sec., it would reach the value of 4 amperes (the amount in the case of the resistance only) in ·5 sec., which is equal to the time-constant $\frac{L}{r}$.

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* It may be mentioned that the curves obtained in this way, like those calculated by the step-by-step method, are not exact owing to the length of the interval. If, however, we add half of this interval to the time-constant, we get the slope of the velocity curve at the middle of the interval, and the results are very close to the theoretical curve.

Now, if both the resistance and inductance are taken into account, we have $V = V_2 + V_3 = rC + LA$, from which

$$A = \frac{V - rC}{L} = \frac{\frac{V}{r} - C}{\frac{L}{r}} = \frac{\hat{C} - C}{T} \quad . \quad . \quad (9b)$$

corresponding to the equation $a = \frac{\hat{v} - v}{T}$ for the inertia and friction combination. We will first work out the rise of current by calculation and afterwards solve it graphically. At the instant of switching on $C = 0$, and thus $rC = 0$ and $A = \frac{V}{L} = 8$ amperes per sec. as before. In a time of ·1 sec., therefore, the current would have grown at this rate to ·8 ampere, and $V_2 = Cr = 25 \times {\cdot}8 = 20$ volts. This leaves $100 - 20 = 80$ volts for V_3 and $A = \frac{V_3}{L} = \frac{80}{12{\cdot}5} = 6{\cdot}4$ amps. per sec. In the next ·1 sec., therefore, the current will increase by ·64 amp. or to $\cdot 8 + \cdot 64 = 1{\cdot}44$ amps. altogether. In this way we get the following table :

TABLE IV.—RISE OF CURRENT IN FIELD MAGNET COIL

$V = 100$, $r = 25\omega$, $L = 12{\cdot}5$ henries.

Time in Secs.	Current Amperes.	$V_2 = Cr$.	$V_3 = V - V_2$.	$A = \frac{V_3}{L}$.
0	0	0	100	8
·1	·8	20	80	6·4
·2	1·44	36	64	5·12
·3	1·95	48·8	51·2	4·10
·4	2·36	59	41	3·28
·5	2·69	&c.	&c.	&c.

In solving the problem graphically we have only to draw a horizontal line representing the ultimate or maximum current of $\hat{C} = \frac{V}{r} = 4$ amperes, and to mark off a length $T = \frac{L}{r} = {\cdot}5$ sec. along it. A sloping line from the origin to this point is then the commencement of the current curve, and if we keep on drawing lines to points ·5 sec. ahead on this line in a similar manner to the procedure in Fig. 26 we get the complete curve. In Fig. 27 is shown the effect of switching the coil on to 100 volts for 2 secs., and of short-circuiting it at that time. The curve of fall of current is obtained by drawing lines from the actual current to points ·5 sec. ahead on the line of ultimate current, which is here, of course, zero.

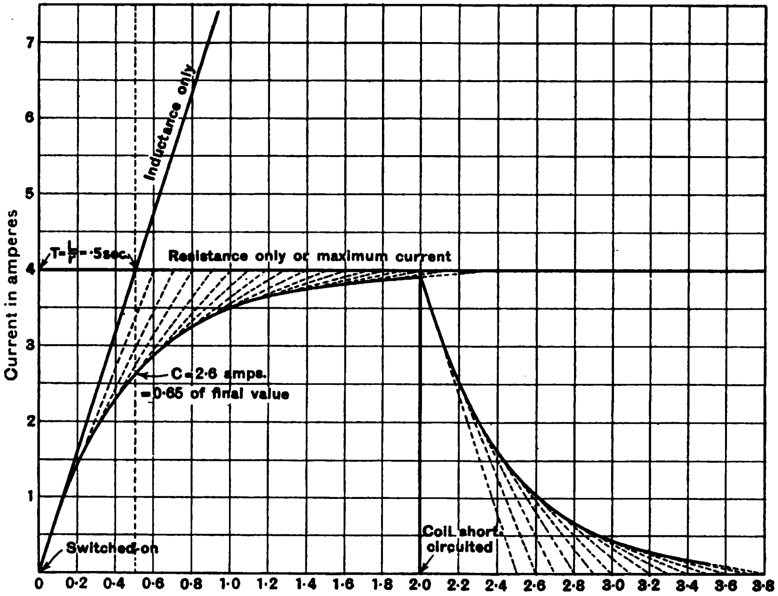


FIG. 27.—Rise and fall of current in field coil. $V = 100$; $r = 25\omega$; $L = 12{\cdot}5$ henries; $T = {\cdot}5$ sec.

Friction and Elasticity.—Let us now return to our motor-boat problem, and suppose that the boat is moored to a fixed object by an elastic rope which has a yield-constant Y, or which yields Y ft. for each pound of force. We will first leave the inertia out of account altogether, and consider what will happen if the engines are started or stopped when this elastic cord is attached and we have the ordinary friction of the boat.

Suppose, at the outset, that we have the elasticity of the rope only to contend with, and that there is no friction. In this case $F = F_1 = \frac{1}{Y}s$, or $s = FY$, or the boat, on the application of the driving force, would immediately move a distance FY forward and then stop. If we plot a curve in which time is measured horizontally, and displacement vertically, this will be represented by the horizontal straight line PQ of constant displacement, Fig. 28. If, on the other hand, we had the boat moving with friction only and unrestrained by the rope,

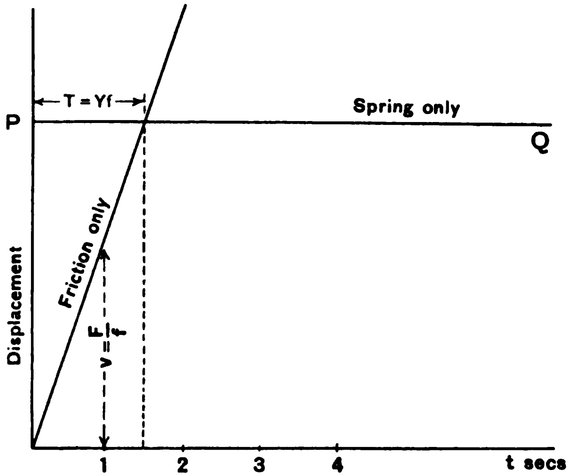


FIG. 28.—Body moving with elasticity only or with friction only.

we should have $F = F_2 = fv$, from which $v = \frac{F}{f}$ as before. But we are now plotting displacement instead of velocity, so that we have $s = vt = \frac{F}{f}t$, or the displacement curve due to friction rises proportionally to the time, just as the velocity curve due to inertia rose in Fig. 23. It will be seen in this case that the displacement due to friction will reach the value due to the elasticity of the cord in a time T, for which $vT = \frac{F}{f}T = FY$, from which $T = Yf$ seconds.

Now consider the elasticity and friction together. The total force F is then made up of the two portions F_1 and F_2 employed in stretching the cord and overcoming the friction, and we therefore have $F = \frac{1}{Y}s + fv$. This gives us the velocity—

$$v = \frac{F - \frac{1}{Y}s}{f} = \frac{YF - s}{Yf}$$

But we have just seen that YF is the total or maximum displacement of the body if it were controlled by the elastic force only, which we will call $\mathfrak{s}$; while $Yf = T$ is the time taken for the displacement when moving against friction only to reach the displacement due to elasticity. This time we will call the time-constant of the combination of elasticity and friction, as it is analogous to the time-constant in the case of the inertia and friction.

The equation of motion of the boat is therefore $v = \frac{YF - s}{Yf} = \frac{\mathfrak{s} - s}{T}$, or the velocity of the boat at any instant is the difference between the maximum or ultimate displacement and the actual displacement, divided by the time-constant. Now if we turn back to the inertia-friction combination we had the equation $a = \frac{\mathfrak{v} - v}{T}$, which is very similar to the last equation, except that it gives the relation between the acceleration and the velocity, instead of that between the velocity and the displacement. But the velocity is the rate at which the displacement changes, and the acceleration is the rate at which the velocity changes. Hence both these equations agree in saying that the rate of change of the displacement or velocity is the amount by which it falls short of its final value divided by the time-constant, and it therefore follows that the displacement increases in the case of the elasticity-friction combination in precisely the same manner as the velocity increases in the friction-inertia combination. The displacement curve can thus be obtained by a graphical method similar to that by which we previously obtained the velocity curve.

To illustrate this we will take the case of the motor-boat propelled with a force of 20 lbs. and having a friction-constant of 2 lbs. per ft. per sec. as before, and imagine it moored by a rope which gives an elastic stretch or extension of ·1 ft. per lb. We thus have $F = 20$, $Y = ·1$, $f = 2$, and we shall for the present neglect the inertia. First assume that there is no friction. Then on applying the driving force we should have a displacement $s = YF = \cdot 1 \times 20$ or 2 ft., and the boat would instantly shoot forward by this amount, and remain in the new position as indicated by the horizontal line in Fig. 29. On the other hand, if friction only were present, we should have $v = \frac{F}{f} = \frac{20}{2} = 10$ ft. per sec. as in the last problem, and the displacement would increase by 10 ft. in each second, or by 1 ft. in each tenth of a second, giving us the sloping line in

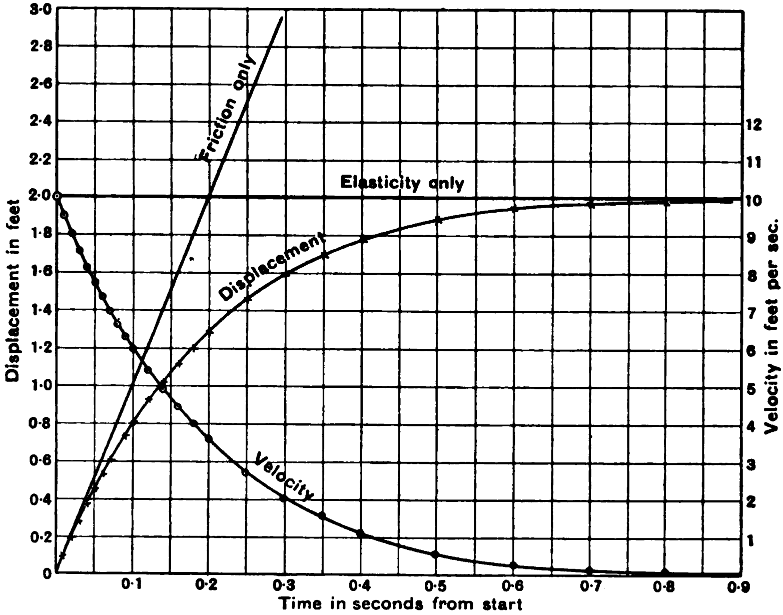


FIG. 29.—Motion of boat with elasticity and friction.
$F = 20$; $Y = \cdot 1$; $f = 2$.

Fig. 29. These two lines cut one another at ·2 sec., which is the same as the constant $T = Yf = \cdot 1 \times 2$ or ·2 sec.

To get the actual displacement and velocity by calculation for the combination of elasticity and friction, we write $F = F_1 + F_2 = \frac{1}{Y}s + fv$, and draw up a table as under. Starting with no displacement, we have $s = 0$ and $v = \frac{F}{f} =$ 10 ft. per sec. just as if there were no elasticity. In ·01 sec., therefore, it would have moved a distance $s = vt = {\cdot}01 \times 10$ or ·1 ft. This would produce an elastic force of $F_1 = \frac{1}{Y}s = \frac{1}{{\cdot}1} \times {\cdot}1$ or 1 lb., leaving a force $F_2 = F - F_1 = 20 - 1$ or 19 lbs. to overcome friction. The velocity would then be reduced to $\frac{F_2}{f} = \frac{19}{2}$ or 9·5 ft. per sec., and the distance moved during the second ·01 sec. = ·01 × 9·5 or ·095 ft., which with the previous displacement of ·1 ft = ·195 ft., producing an elastic drag of $F_1 = \frac{{\cdot}195}{{\cdot}1} = 1{\cdot}95$ lbs. The following table is made out in this way.

TABLE V.—MOTION OF BOAT MOORED BY ELASTIC CORD

$F = 20$ lbs., $Y = {\cdot}1, f = 2$.

Time from start in Secs.	Displacement s feet.	$F_1 = \frac{1}{Y}s$.	$F_2 = F - F_1$.	$v = \frac{F_2}{f}$.
0	0	0	20	10
·01	·1	1	19	9·5
·02	·195	1·95	18·05	9·03
·03	·2853	2·853	17·15	8·58
·04	·3711	3·711	16·29	8·15
·05	·4526	4·526	15·47	7·74
·06	·5399	5·399	14·7	7·35
·07	·6035	6·035	13·96	6·98
·08	·6733	6·733	13·27	6·63
·09	·7396	7·396	12·60	6·30
·10	·8026	8·026	11·97	5·99
·12	·9223	9·223	10·78	5·39
·14	1·010	10·10	9·90	4·95
·16	1·109	11·09	8·91	4·46
·18	1·198	11·98	8·02	4·01
·20	1·278	12·78	7·22	3·61
·25	1·458	14·58	5·42	2·71
·30	1·594	15·94	4·06	2·03
·35	1·696	16·96	3·04	1·57
·40	1·775	17·75	2·25	1·13
·50	1·888	18·88	1·12	·56
·60	1·944	19·44	·56	·28
·70	1·972	19·72	·28	·14
·80	1·986	19·86	·14	·07

As indicated by the above reasoning, the displacement curve rises in the same way as the velocity rose in the case of the combination of inertia and friction. It rises at first just as if the friction alone operated, then bends over and gradually approaches the value which it would attain if elasticity only existed. In a time equal to the time-constant the displacement is 1·28 ft., which, when divided by the maximum displacement of 2 ft., gives ·64 of the final value. This fraction should, as we have before said, be ·634. In one second the displacement has nearly reached its final value. The velocity rises instantly to a maximum at first just as if the spring had not been there, and gradually falls as the elastic force of the spring more nearly attains to equality with the driving force. It is evident that the curve could have been obtained without calculation by drawing lines to points on the maximum displacement line which are a time T ahead of the actual times, as was done in the case of the friction-inertia combination in Fig. 27. The recoil of the boat when the engines are stopped can be similarly shown.

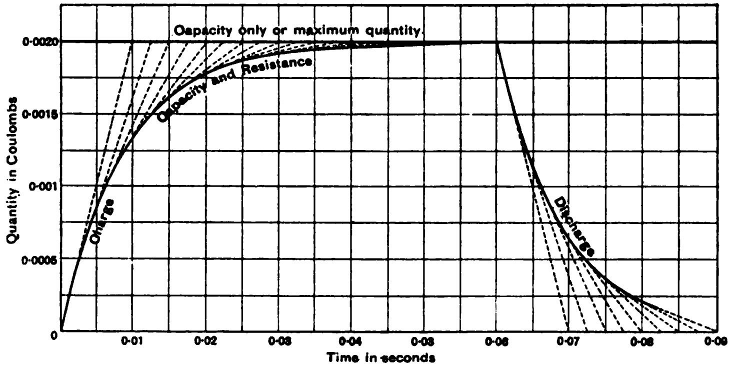


FIG. 30.—Charge and discharge curve of condenser.

Condenser Charge and Discharge.—Since a condenser is the electrical equivalent of a spring, the application of a force to a combination of elasticity and friction, with which we have just dealt, is equivalent to the application of an equal voltage to a combination of capacity and resistance of corresponding amounts. In fact, if we consider the charging of a ·1-farad condenser by an E.M.F. of 20 volts through a resistance of 2 ohms, the curves giving the quantity in coulombs and the current in amperes would then be identical with those just obtained for the displacement in feet, and the velocity in ft. per sec. If, on the other hand, a more practical case is taken, say, a 10-microfarad condenser charged through a resistance of 1000 ohms, with an E.M.F. of 200 volts, we have $K =$

$\frac{10}{1,000,000} = 10^{-5}$ farads, $r = 1000$ ohms, and $V = 200$ volts. Then $V = V_1 + V_2 = \frac{1}{K}Q + rC$, from which $C = \frac{KV - Q}{Kr} = \frac{\hat{Q} - Q}{T}$ where $\hat{Q} = KV$ is the maximum charge corresponding to $\hat{s} = YF$, and T is the time-constant Kr corresponding to Yf secs. Hence $\hat{Q} = 10^{-5} \times 200 = \cdot 002$ coulombs, and $T = 10^{-5} \times 1000 = \cdot 01$ sec., which is the time for the charge to reach ·634 or 63·4 °/$_\text{o}$ of its final value.

The following table is worked out in a similar way to the mechanical problem last treated, and needs no explanation, while the curves of charge and discharge (Fig. 30) are obtained by the graphical method, each part of the curve being drawn as if it were a portion of a straight line directed to a point $T = \cdot 01$ sec. ahead.

TABLE VI.—CHARGE OF CONDENSER THROUGH RESISTANCE

$K = 10$ mfds. $= 10^{-5}$ farads, $r = 1000\omega$.

$E = 200$ volts, $\hat{C} = \frac{E}{r} = \cdot 2$ amp., $T = Kr = \cdot 01$ sec.

Time t Secs.	Quantity Q Coulombs.	P.D. for Condenser $V_1 = \frac{1}{K}Q$ Volts.	P.D. for Resistance $V_2 = E - V_1$ Volts.	Current $C = \frac{V_2}{r}$ Amperes.
0	·0	0	200	·2
·001	·00020	20	180	·18
·002	·00038	38	162	·162
&c.	&c.	&c.	&c.	&c.

As a matter of fact, this case is one of the easiest to show experimentally. If we have an electrostatic voltmeter connected to the condenser terminals, $V_1 = \frac{1}{K}Q$, or $Q = KV_1$, so that the quantity at every instant is given by multiplying the voltmeter reading by the capacity of the condenser. If we take a 50-microfarad condenser and charge and discharge it through a resistance of a megohm, we have a time-constant $T = Kr = \frac{50}{10^6} \times 10^6$ or 50 secs., and the slow rise and fall of the voltmeter reading can be distinctly seen. Fig. 31 shows the actual observations obtained with such a combination on a 100-volt circuit. The current curve was obtained by connecting a second electrostatic voltmeter across the terminals of the resistance which read $V_2 = Cr$; thence $C = \frac{V_2}{r}$, the voltmeter being of infinite resistance. As the resistance in series was one megohm, each volt of P.D. across it represents one microampere of current.

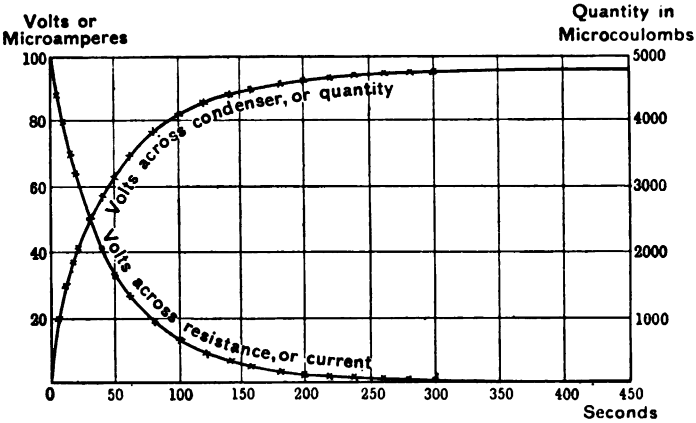


FIG. 31.—Experimental curves of charge and discharge of condenser.

Planimeter for drawing Displacement or Current Curves.—It will have been noticed that in four cases—(a) the motion of a body with friction and inertia, (b) the change of current with resistance and inductance, (c) the displacement of a body with elasticity and friction, and (d) the charge of a condenser with resistance in circuit, the law of motion is $\dot{y} = \frac{\hat{y} - y}{T}$, where y represents the displacement, velocity, quantity or current, $\hat{y}$ its final value, $\dot{y}$ its rate of change, and T the time-constant. We have also seen that this means that the curve of y with time rises or falls as if it was always directed to a point on the line representing the final value of y, a time T ahead. This is the way in which the curves in Figs. 26 and 27 were obtained. But it is quite simple to make an instrument which will do this automatically and trace out the curve for us. In Fig. 32 is shown a piece of metal in the form of the letter **T**, the vertical arm having a slot in it along which a block B_1 slides freely. On this block is mounted a swivelling crutch C in which is pivoted a sharp-edged wheel *W*, which rolls on the paper and marks off the required curve. A light rod *R* is attached to the crutch *C* in such a manner that the edge of the wheel is always in line with the rod, and this rod passes freely through a hole in a swivelling brass block B_2, which can be clamped in a groove in the other arm of the **T** and at a distance from the centre line of the other groove equal to the time-constant. To facilitate this a scale is provided on the horizontal arm. Then if we draw this apparatus along a sheet of paper,

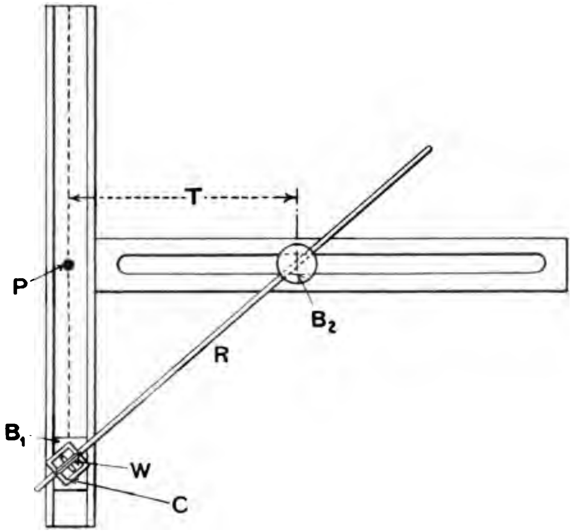


FIG. 32.—Planimeter for drawing displacement or current curves. *View from below.*

always keeping it parallel to itself, and so that the point *P*, where the centre lines of the two slots meet, is moved along the line of final value of y, or $\hat{y}$, the wheel will trace out the actual curve of y. For at any instant the wheel is at a certain distance from $P = \hat{y} - y$, and since it has a sharp edge it can only roll in the direction of the rod which points to the block B_2, which is a fixed distance *T* ahead. Fig. 33 shows the instrument provided with a linkwork which keeps it always parallel to itself, on a drawing board, and actually drawing some curves. It is clear that $\hat{y}$ will change if the driving force or voltage changes, and the wheel will always trace out the corresponding change in y. The curves *A* in Fig. 33 are for the starting and stopping of a current, the upper curve being for a small and the lower for a large time-constant. In the actual instrument the tracing point P is mounted at a definite distance from the

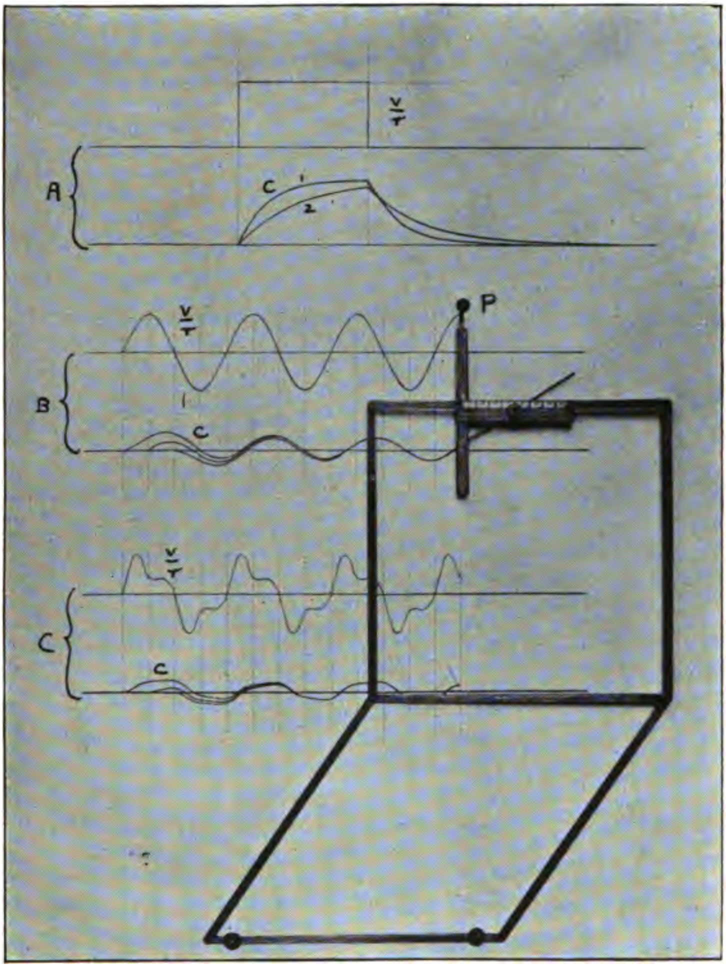


FIG. 33.—Planimeter and curves: (*A*) for starting and stopping a direct current, (*B*) for switching on an A.C. circuit with sinusoidal voltage, (*C*) ditto with non-sinusoidal voltage.

intersection of the slots in order to permit the free motion of the wheel, and in consequence the current curve is drawn on an axis at that distance from the axis of the original curve.

Combinations of Elasticity, Friction, and Inertia.—We can now finally treat the complete problem when the three properties, elasticity, friction, and inertia, are to be combined together. This, with the corresponding electrical combination of capacity, resistance, and inductance, is so important that it will receive considerable attention later, and only a few words will be devoted to it here.

The force in the case of such a combination has evidently to perform three operations: the stretching of the spring, the overcoming of the friction, and the acceleration of the mass. Hence the total force $F = F_1 + F_2 + F_3 = \frac{1}{Y}s + fv + ma$. We will leave the discussion of this equation till later, and simply take the case of our motor-boat with a friction-constant of 10 lbs. per ft. per sec., mass of 300 engineers' units, and elastic control with a yield-constant of ·1 ft. per lb., and consider what will happen when the engines are suddenly started. At the moment of starting $s = 0$ and $v = 0$, from which $F = ma$, or $a = \frac{F}{m} = \frac{20}{300} = {\cdot}0667$, and the boat starts with the same acceleration as if there were no frictional or elastic control. In one second the velocity of the boat will therefore have reached ·0667 ft. per sec., but the displacement s will be the average velocity multiplied by the time, or $\frac{0 + {\cdot}0667}{2} \times 1 = {\cdot}0333$ ft. To produce the velocity of ·0667 ft. per sec. with a friction-constant f of 10 lbs. requires a force $F_2 = fv = 10 \times {\cdot}0667 = {\cdot}667$ lb., and to displace the boat ·0333 ft. with a yield-constant $Y = {\cdot}1$ requires a force $F_1 = \frac{1}{Y}s = 10s = 10 \times {\cdot}0333 = {\cdot}333$ lb. Of the total propelling force of 20 lbs., therefore, a portion $F_1 = {\cdot}333$ lb. is employed in stretching the cord, and a portion $F_2 = {\cdot}667$ lb. in overcoming the friction, leaving an amount $F_3 = F - F_1 - F_2 = 20 - {\cdot}333 - {\cdot}667$ or 19 lbs., to accelerate the mass. The acceleration will therefore be $a = \frac{F_3}{m} = \frac{19}{300}$ or ·0633 ft. per sec. Hence in the next second the velocity will have increased by ·0633 ft. per sec., and this, added to the velocity at the end of the first second, viz. ·0667 ft. per sec., gives ·130 ft. per sec. at the end of

2 secs. from the start. The average velocity during the second second will be $\frac{\cdot 0667 + \cdot 130}{2} = \cdot 0984$ ft. per sec., and this would produce a displacement during that second of ·0984 ft., which, added to the displacement at the end of the first second, viz. ·0333 ft., gives a total displacement of

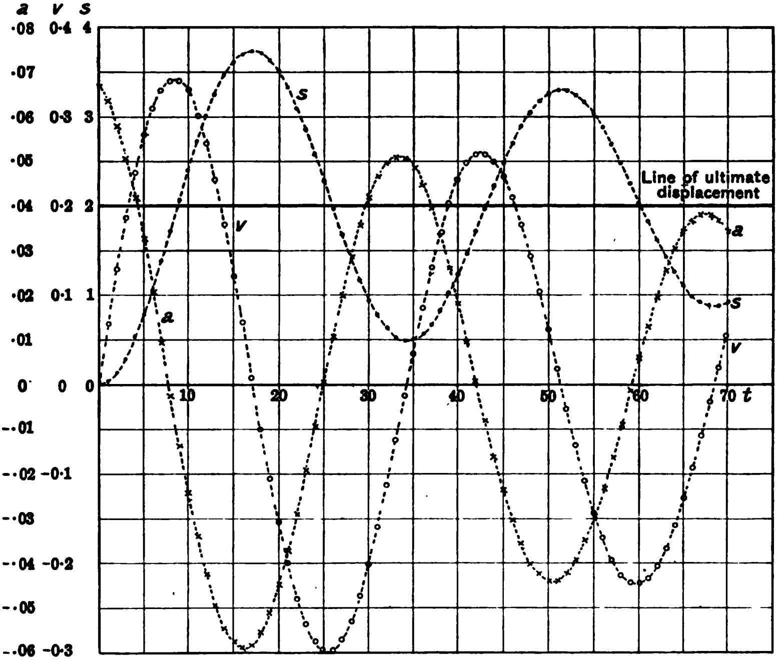


FIG. 34.—Displacement, velocity, and acceleration curve for boat.
$F = 20$; $Y = \cdot 1$; $f = 10$; $m = 300$.

·0984 + ·0333 = ·1317 ft. Multiplying this new velocity and displacement by f and by $\frac{1}{Y}$ respectively, we have the forces F_2 and F_1 expended in overcoming friction and elasticity, and thus obtain the new acceleration, and so on. In Table VII this process has been carried out at length, and the student should carefully follow the working, especially where, as will be seen, the acceleration and other quantities change sign. In Fig. 34 the values of the displacement, velocity, and acceleration of the boat are plotted; we see that instead of it moving forward quickly at

TABLE VII.—MOTION OF BOAT WITH ELASTICITY, FRICTION, AND MASS

$$F = 20 \text{ lbs.},\ Y = \cdot 1,\ f = 10,\ m = 300,$$
$$20 = 10s + 10v + 300a.$$

t.	v.	s.	F_1.	F_2.	F_3.	a.
0	0	0	0	0	20	·0667
1	·0667	·0333	·333	·667	19	·0633
2	·1300	·1317	1·317	1·300	17·383	·0579
3	·1879	·2907	2·907	1·879	15·214	·0507
4	·2386	·5040	5·040	2·386	12·574	·0419
5	·2805	·7636	7·636	2·805	9·559	·0329
6	·3134	1·0605	10·605	3·134	6·261	·0209
7	·3343	1·3843	13·843	3·343	2·814	·0094
8	·3437	1·7230	17·230	3·437	– ·667	– ·0022
9	·3415	2·0656	20·656	3·415	– 4·071	– ·0136
10	·3279	2·4003	24·003	3·279	– 7·282	– ·0243
11	·3036	2·7161	27·161	3·036	– 10·197	– ·0340
12	·2696	3·0027	30·027	2·696	– 12·723	– ·0424
13	·2272	3·2511	32·511	2·272	– 14·783	– ·0493
14	·1779	3·4537	34·537	1·779	– 16·316	– ·0544
15	·1235	3·6044	36·044	1·235	– 17·279	– ·0576
16	·0659	3·6991	36·991	·659	– 17·650	– ·0588
17	·0071	3·7356	37·356	·071	– 17·427	– ·0581
18	– ·0510	3·7137	37·137	– ·510	– 16·627	– ·0554
19	– ·1064	3·6350	36·350	– 1·064	– 15·286	– ·0510
20	– ·1574	3·5031	35·031	– 1·574	– 13·457	– ·0449
21	– ·2023	3·3233	33·233	– 2·023	– 11·210	– ·0374
22	– ·2397	3·1032	31·032	– 2·397	– 8·635	– ·0288
23	– ·2685	2·8491	28·491	– 2·685	– 5·806	– ·0194
24	– ·2879	2·5709	25·709	– 2·879	– 2·830	– ·0094
25	– ·2973	2·2783	22·783	– 2·973	·190	·0006
26	– ·2967	1·9813	19·813	– 2·967	3·154	·0105
27	– ·2862	1·6899	16·899	– 2·862	5·963	·0199
28	– ·2663	1·4136	14·136	– 2·663	8·527	·0284
29	– ·2379	1·1615	11·615	– 2·379	10·764	·0359
30	– ·2020	·9415	9·415	– 2·020	12·605	·0420
31	– ·1600	·7605	7·605	– 1·600	13·995	·0466
32	– ·1134	·6238	6·238	– 1·134	14·896	·0497
33	– ·0637	·5352	5·352	– ·637	15·285	·0510
34	– ·0127	·4970	4·970	– ·127	15·157	·0505
35	·0378	·5096	5·096	·378	14·526	·0484
36	·0862	·5716	5·716	·862	13·422	·0447

TABLE VIII.—MOTION OF BOAT WITH ELASTICITY, FRICTION, AND MASS

$$F = 20,\ Y = \cdot 1,\ f = 50,\ m = 300,$$
$$20 = 10s + 50v + 300a.$$

t.	v.	s.	F_1.	F_2.	F_3.	a.
0	0	0	0	0	20	·0667
1	·0667	·0333	·333	3·333	16·333	·0544
2	·1211	·1272	1·272	6·055	12·673	·0422
3	·1633	·2694	2·694	8·165	9·141	·0305
4	·1938	·4479	4·479	9·690	5·831	·0194
5	·2132	·6514	6·514	10·660	2·826	·0094
6	·2226	·8693	8·693	11·130	·177	·0006
7	·2232	1·0922	10·922	11·160	-2·082	-·0069
8	·2163	1·3119	13·119	10·815	-3·934	-·0131
9	·2032	1·5216	15·216	10·160	-5·376	-·0179
10	·1853	1·7158	17·158	9·265	-6·423	-·0214
11	·1639	1·8904	18·904	8·195	-7·099	-·0237
12	·1402	2·0425	20·425	7·010	-7·435	-·0248
13	·1154	2·1703	21·703	5·770	-7·473	-·0249
14	·0905	2·2732	22·732	4·525	-7·257	-·0242
15	·0663	2·3516	23·516	3·315	-6·831	-·0228
16	·0435	2·4065	24·965	2·175	-6·240	-·0208
17	·0227	2·4396	24·396	1·135	-5·531	-·0184
18	·0043	2·4531	24·531	·215	-4·746	-·0158
19	-·0115	2·4495	24·495	-·575	-3·920	-·0131
20	-·0246	2·4315	24·315	-1·230	-3·085	-·0103
21	-·0349	2·4018	24·018	-1·745	-2·273	-·0076
22	-·0425	2·3631	23·631	-2·125	-1·506	-·0050
23	-·0475	2·3181	23·181	-2·375	-·806	-·0027
24	-·0502	2·2693	22·693	-2·510	-·183	-·0006
25	-·0508	2·2188	22·188	-2·540	+·352	+·0012
26	-·0496	2·1686	21·686	-2·480	·794	·0026
27	-·0470	2·1203	21·203	-2·350	1·147	·0038
28	-·0432	2·0752	20·752	-2·160	1·408	·0047
29	-·0385	2·0343	20·343	-1·925	1·582	·0053
30	-·0332	1·9984	19·984	-1·660	1·676	·0056
31	-·0276	1·9680	19·680	-1·380	1·700	·0057
32	-·0219	1·9432	19·432	-1·095	1·663	·0055
33	-·0164	1·9241	19·241	-·820	1·579	·0053
34	-·0111	1·9103	19·103	-·555	1·452	·0048
35	-·0063	1·9016	19·016	-·315	1·299	·0043
36	-·0020	1·8975	18·975	-·100	1·125	·0038
37	+·0018	1·8974	18·974	·090	·936	·0031
38	·0049	1·9008	19·008	·245	·747	·0025
39	·0074	1·9069	19·069	·370	·561	·0019
40	·0093	1·9152	19·152	·465	·383	·0013

first and more slowly afterwards to a definite displacement of 2 ft., as was the case when there was no inertia, it now gradually increases its speed and displacement, and actually overshoots its mark, reaching a maximum displacement of 3·75 ft., afterwards getting up speed in the opposite direction, and so on, and these reversals or oscillations will repeat several times. Table VIII gives the calculations for $F = 20$, $Y = \cdot 1$, $f = 50$, and $m = 300$. In Fig. 35 displacement curves are shown obtained in the same manner, for $f = 10$, $f = 20$, $f = 50$, and $f = 110$, and it will be seen that in the first two cases the boat oscillates backwards and forwards, but that as the friction is increased the oscillations die away more rapidly until when $f = 110$ there is no oscillation at all. A capital example of this is given by an ordinary cylindrical spring attached by a hook to a board as in Fig. 36. If the board is held quite upright, and a weight suddenly dropped on the hook, the spring will stretch and oscillate up and down for some time. This is an example of elasticity and inertia with practically no friction. If, however, the same experiment is performed when the board is inclined somewhat backward, as in Fig. 37, the weight will rub against the board during its motion, so that the friction is increased (although it is not quite the right sort of friction, being between two solid substances), and it will be found that the oscillations die away

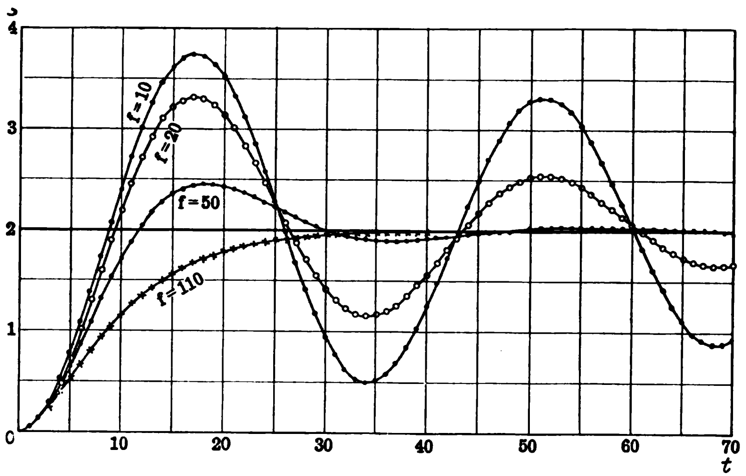


FIG. 35.—Displacement curves for boat.
$F = 20$; $Y = \cdot 1$; $m = 300$; $f = 10, 20, 50,$ or 110.

more or less rapidly, the more or less the board is inclined to the vertical.

Another way of showing this is by employing a steel strip as a spring, and fixing a mass on the end of it. On suddenly applying a force by means of a weight and cord, as shown in Fig. 38*a*, the spring will bend to one side and oscillate, and if we attach to it a brush dipped in ink, arranged to touch against a sheet of paper which is steadily drawn through a slot in the board, we get traces as shown in Fig. 38*b*, which are of the same form as that we have obtained by calculation. If the load is suddenly removed the spring flies back and oscillates about its position of rest.

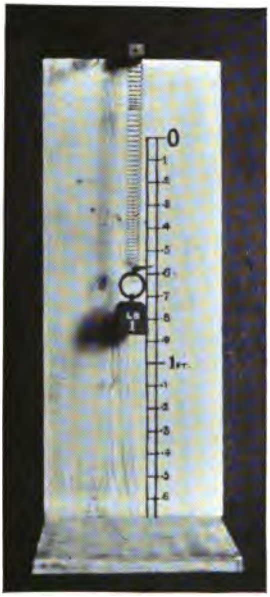


FIG. 36.—Spring and weight upright.

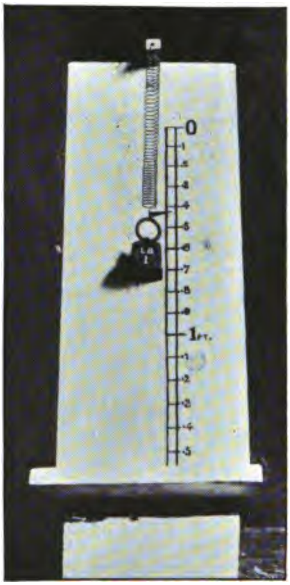


FIG. 37.—Spring and weight inclined.

Combinations of Capacity, Resistance, and Inductance.—If the reader has followed the analogies up to this point, he will see immediately that if a condenser is joined in series with a resistance and inductance the total voltage required to charge the condenser, force the current through the resistance, and accelerate it against the inductance must be—

$$V = V_1 + V_2 + V_3 = \frac{1}{K}Q + rC + LA$$

which exactly resembles the equation—

$$F = F_1 + F_2 + F_3 = \frac{1}{Y}s + fv + ma$$

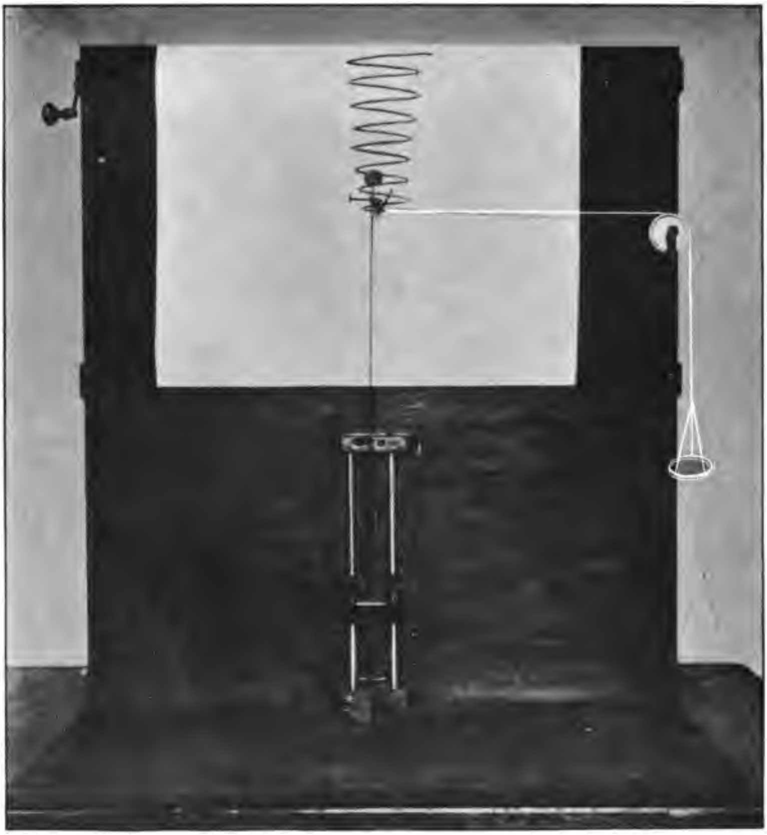


FIG. 38*a*.—Spring and weight model.

for the combination of elasticity, friction, and inertia.—The quantity, current, and electrical acceleration must therefore vary in an absolutely similar manner to the displacement, velocity, and acceleration in the mechanical case. In Fig. 39 are given experimental curves taken with an oscillograph of the discharge of a condenser in series with an inductive coil and with two different resistances, which shows that this is true. We might have worked out a table and curves for the electrical case precisely as we did for the mechanical case in Table V, but this can be easily done by the reader, and the whole subject of electrical oscillations is so important as to require further consideration later. The student should, however, work out at least one such exercise himself, as, for example, the charging of a 10-mfd. condenser in series with a coil of ·05 henries inductance and 50 ohms resistance, with an E.M.F. of 100 or 200 volts, and its discharge. When this has been done there ought to be nothing in ordinary alternate current work which will be found difficult to understand.

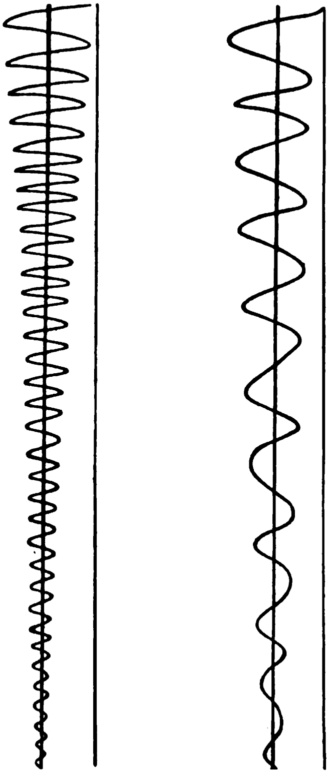


FIG. 38*b*.—Curves obtained from spring model with small and large inertia.

All the conclusions we have arrived at in this chapter can be approximately verified with the aid of the truck model. In Fig. 40 the truck is shown carrying both the sail and a weight, while the cord is attached to the spring. Under these circumstances, if a driving force is suddenly applied, oscillatory motions like those in Fig. 35 are obtained. If the spring is detached from the cord, we have a combination of friction and inertia only, and the truck will get up speed on the application

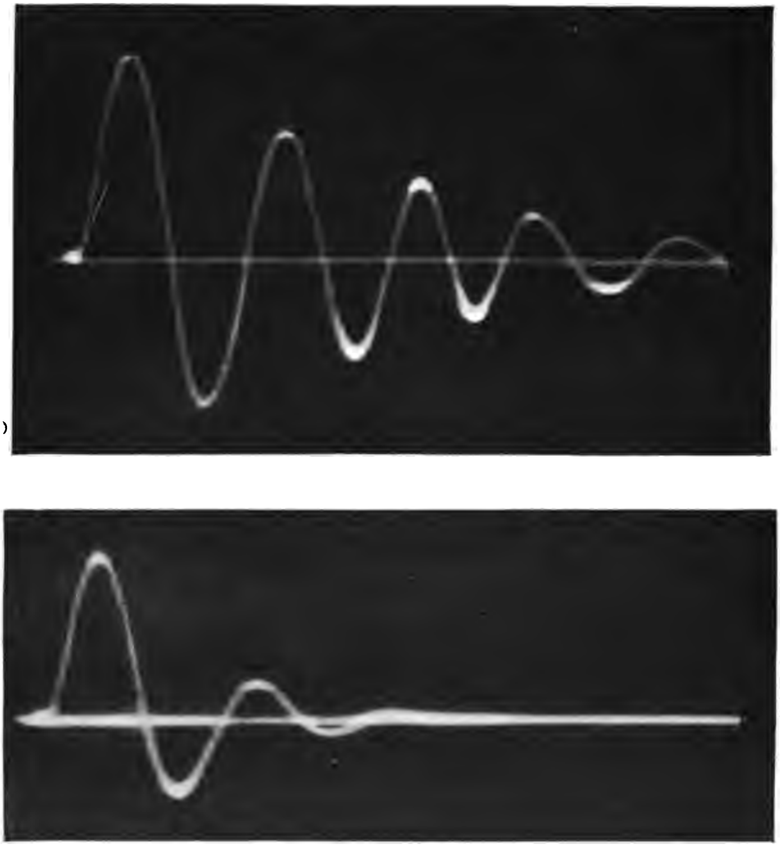


FIG. 39.—Oscillogram of condenser discharging, with low and high resistance respectively.

of a driving force, in the same way as the boat in Fig. 24. On the other hand, if the spring is attached to the cord again, and

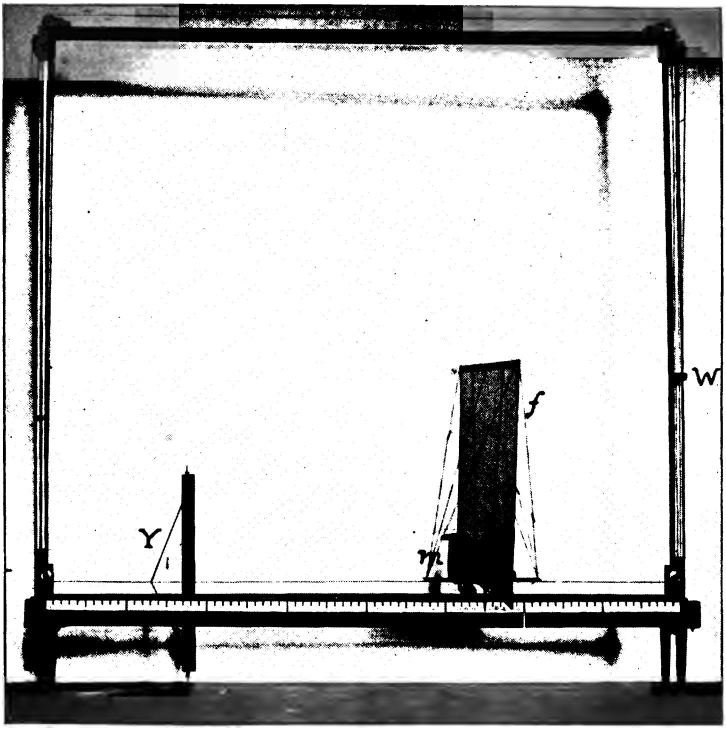


FIG. 40.—Mechanical model of circuit with capacity, resistance, and inductance.

the weight removed, the combination is one of elasticity and friction, and it behaves under the action of a force, as shown in the curve of Fig. 29.

PART II

SIMPLE HARMONIC MOTION

CHAPTER VI

PROPERTIES OF SIMPLE HARMONIC MOTION

We have now arrived at the stage where, if we know the displacement, velocity, and acceleration of a body, we can determine the force required to produce this motion; and similarly if we know the quantity, current, and electrical acceleration in a circuit, we are able to determine the applied voltage. The next stage is evidently as follows: Given the motion of a body, or the variation of a current, how can we find the displacement, velocity, and acceleration, or the quantity, current, and electrical acceleration, in order to be able to calculate the required forces?

Obviously these quantities are, as a rule, quite independent of one another. A tramcar may have any combination of displacement, velocity, and acceleration; but no one of these quantities gives any clue to the others. In some few cases, however, this is not so. For instance, if a body swings backwards and forwards or vibrates, so as to repeat the same changes over and over again, its displacement, velocity, and acceleration are dependent one upon the other, and if one is known the others can be found.

Now, there are various types of such oscillating or vibrational motion. Perhaps the simplest at first sight would be that of the bed of a planing machine, in which there is a steady motion in one direction along the whole of one stroke, followed by a sudden reversal and a steady traverse in the other direction. But this is a very unusual and forced motion, and the protesting shriek of the belts at the moment of reversal bears witness to its forced character. Let us take a few cases of bodies vibrating more or less freely or naturally. These are:

A pendulum.
The balance wheel of a watch.
A weight on the end of a spring.
The oscillation of water in a tube.
The motion of a cork on a rippling pond.
A boat on the sea.
A vibrating tuning fork.
A stretched string.

All these and many others are natural vibrations, and would repeat themselves indefinitely if it were not for friction, which makes them die away more or less gradually.

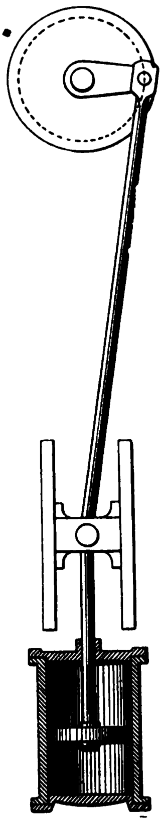
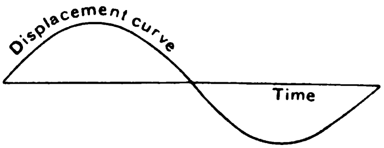


FIG. 41.
Simple engine.



There is, however, another very familiar case of a uniform reciprocating motion which can be most easily studied—the motion of a steam-engine piston, when its crank is steadily rotating. In Fig. 41 we have a simple form of engine, in which the cylinder is vertically below the crank shaft, and in which the piston rod is connected to the crank by means of a very long connecting rod. As the crank rotates the piston moves up and down, and it is clear that the piston reaches the top or the bottom of the stroke when the crank is vertically upwards or downwards. When the crank is horizontal as shown in the illustration, the piston will be practically in the middle of the stroke, and would be exactly so if the connecting rod were so long that it was never very oblique to the vertical centre line. A moment's consideration will show that in this case the position of the piston simply depends upon the height of the crank pin; and that the displacement of the piston above or below its mid-position is equal to the height of the crank pin above or its depth below the horizontal line through the centre of the crank shaft. In Fig. 42 we have shown diagrammatically a crank with a radius of 1 ft. which we will suppose to be rotating uniformly at the rate of 125 revs. per min. A whole revolution will thus be performed in $\frac{60}{125}$ = ·48 sec., and if the circumference of the crank circle is divided into 24 equal parts, the crank will move over each of these in $\frac{\cdot 48}{24}$ = ·02 sec. We can then readily show how the position of the piston varies from time to time by plotting a curve in the following manner. Along a horizontal line *AB* mark off equal intervals of time, ·02, ·04, ·06 sec., &c. If we start at time 0 with the crank horizontal, then the piston will be in its central position or its displacement from that position will be nil. At a time ·02 sec. later the crank will have moved to the position *O*1, and the piston will have been displaced upwards by an amount practically equal to the height of the point 1 above the horizontal. If we set up this height at ·02

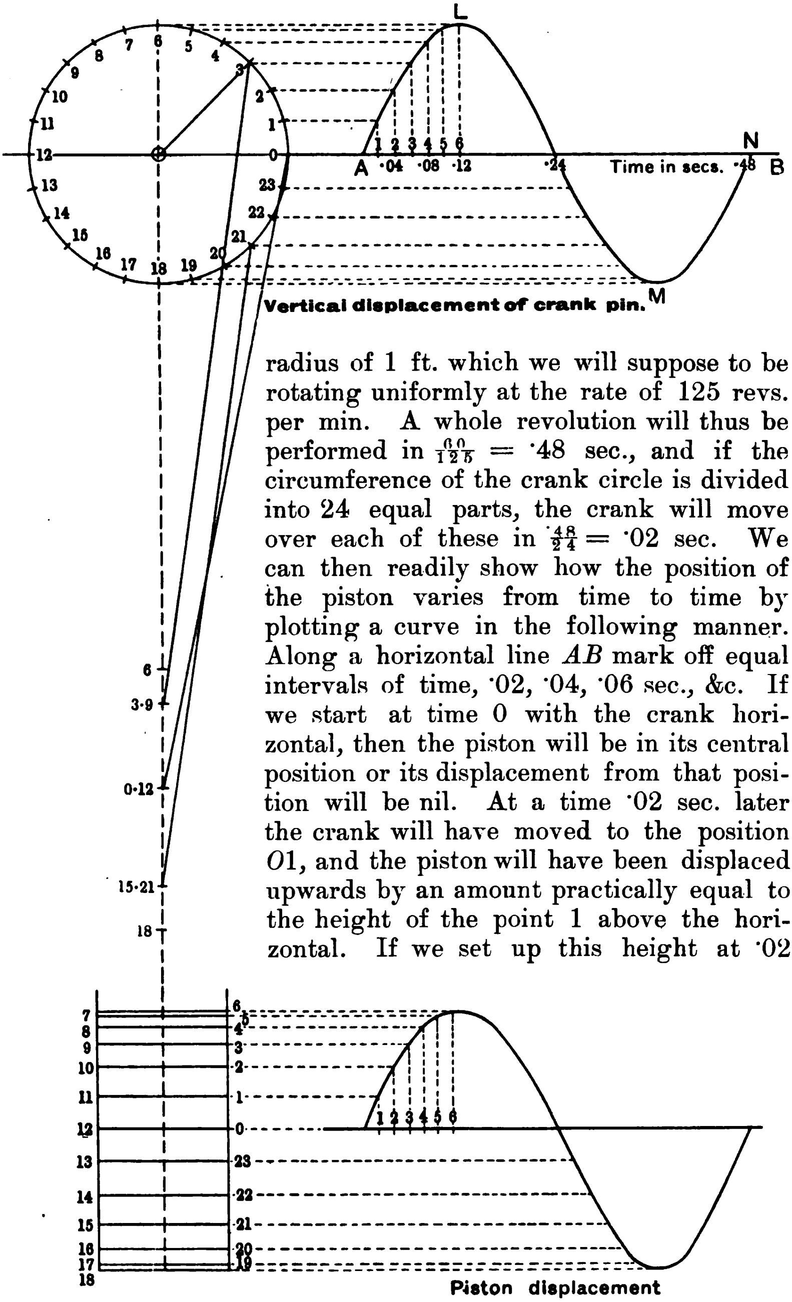


FIG. 42.—Motion of engine piston.

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sec. we have the first point on the displacement curve. At ·04 sec. from the start the crank will be at the position *O*4, and its height projected across to the vertical line at ·04 sec. gives the second point. By repeating this process for each point round the circle we arrive at the curve *ALMN*, which shows us at a glance where the piston is at every instant. This curve would evidently go on repeating itself for each revolution of the crank. As a matter of fact, the displacement of the piston is not quite strictly represented by this curve owing to the obliquity of the connecting rod, but by measuring off equal lengths representing the connecting rod from the crank pin to the axis of the piston, we get the exact positions of the piston, and these give the curve shown in the lower part of the diagram which is practically indistinguishable from the upper one, and which would be absolutely identical with it if the connecting rod were infinitely long.

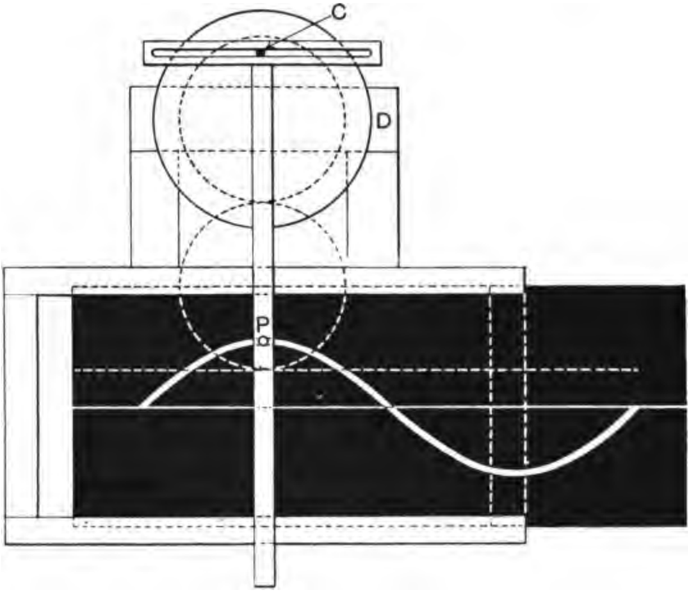

FIG. 43*a*.—Skeleton engine model for drawing simple harmonic curves.

In order to show exactly the way in which this curve is produced the author has had the apparatus shown in Fig. 43 made up. It consists of a disc or flywheel *D*, at any point of the surface of which a crank pin *C* can be fixed. The connecting rod is replaced by a vertical sliding rod with a horizontal slot at the top in which the crank pin moves, and the piston by a clip holding a chalk or pencil P. Behind this pencil a rectangular board is fitted which slides in horizontal grooves, and is so geared to the flywheel or crank that it moves *backwards* by equal intervals as the crank is turned through equal angles. The result is therefore the same as if the piston or pencil moved up and down and at the same time moved *forwards* a distance proportional to the angle or time, and the pencil marks out the displacement curve automatically on the board, like an indicator diagram. The great use of this model will appear later.

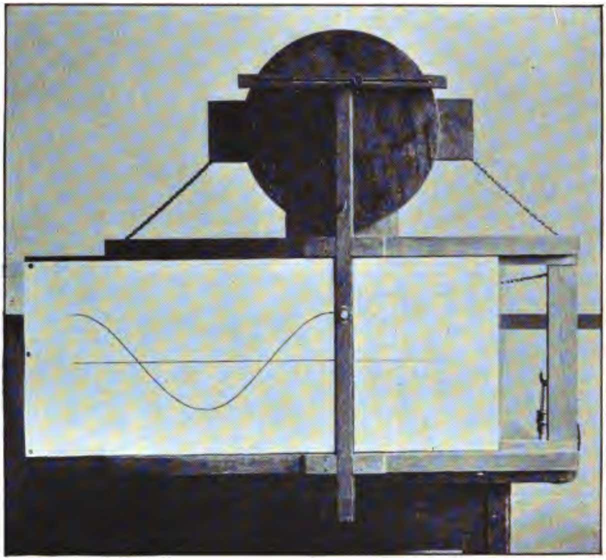

FIG. 43b.—Skeleton engine model for drawing simple harmonic curves.

Now if we take our pendulum, which was our first example of a natural vibration, and attach a brush charged with ink to it, we can let it swing while we pull a strip of paper steadily forwards underneath it, as in Fig. 44; a curve is thus obtained which, when reduced to the same scale, can hardly be distinguished from that obtained in Fig. 42. Similarly, if we take a tuning fork and attach a needle to one of its prongs, and draw it over a piece of smoked glass while it is vibrating, we get a curve of precisely similar form (Fig. 45). A section through the ripples on a pond would have the same outline, and practically all simple natural vibrations have the same kind of motion, which is called Simple Harmonic Motion. The origin of the name is obvious, as it is the motion which every sounding body that gives out a simple or pure tone executes, but we shall find it most simple and useful to think of it as the motion of the piston of an engine when the crank uniformly rotates, and, in fact, the scientific definition of simple harmonic motion is practically equivalent to this.

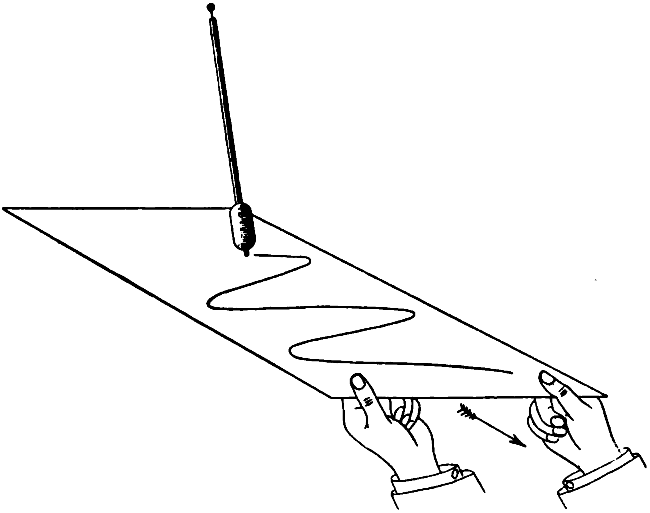

FIG. 44.—Motion of pendulum.

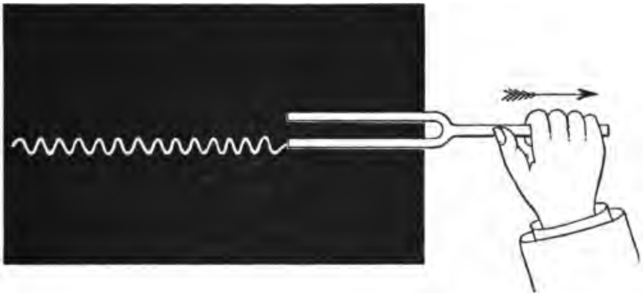

FIG. 45.—Trace of tuning fork.

Definition of Simple Harmonic Motion.—*A simple harmonic motion (S.H.M.) is the motion of a point on the diameter of a circle which is the projection on that diameter of a point uniformly rotating round the circle.*

In Fig. 46 we have a circle in which the point *C* moves uniformly round the circle like the crank pin of a uniformly rotating crank. If we take any diameter *AB* of the circle, and drop a perpendicular *CP* from the point *C* on the diameter *AB*, then as *C* moves uniformly round the circle the foot of the perpendicular *P* will move up and down along the line *AB* with a simple harmonic motion. This is obviously exactly the same motion as that of the piston of our engine or of the pencil in our model (Fig. 43).

Displacement, Velocity, and Acceleration in Simple Harmonic Motion.—As we indicated above, whenever a body oscillates or moves in such a way that it continually repeats the same variations of position, there is some determinable relation between its displacement, velocity, and acceleration. If we can find this relation in the case of a simple harmonic motion, we shall be able to find the force required to make a body move in this manner or the voltage required to produce a simple harmonically varying current. This can be very readily done by studying our engine.

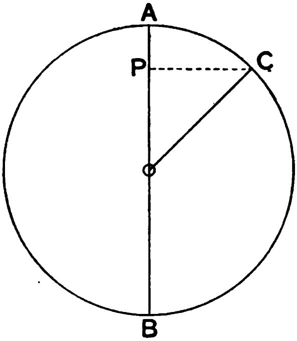

FIG. 46.—Method of deriving simple harmonic motion.

Suppose, as in Fig. 47, we have a crank *OC* of radius *r* ft. rotating with a speed of *n* revs. per sec. If this were attached to a piston through an infinitely long connecting rod, the displacement of the piston from the centre of the stroke would be represented by the vertical height $CD = s$, and it would vary in a simple harmonic manner. As *OCD* is a right-angled triangle, $s = r \sin \theta$ where θ is the angle which the crank makes with the horizontal. Since the angle turned through by the crank in one revolution is 360° or 2π radians, the total angle of rotation per second or the *angular velocity* of the crank is $2\pi n$, which is generally represented by the letter p. In any time t seconds the angle of rotation $\theta = 2\pi nt = pt$, so that $s = r \sin pt$ is the equation which expresses the relation of the displacement s to the time. The displacement therefore depends upon the sine of an angle which is proportional to the time, whence the variation is often spoken of as being *sinusoidal,* and the displacement curve as a *sine curve.* Every simple harmonic motion can therefore be represented by a sine curve.

It is evident that the maximum value which s can have and which we shall denote by $\mathscr{s}$, is equal to the radius r. Hence—

$$s = r \sin pt = \mathscr{s} \sin pt \quad . \quad . \quad . \quad . \quad (10)$$

We next want to know what will be the velocity of the piston at any instant. For this purpose we should naturally start by finding the velocity of the crank pin *C*, to which the piston is connected. Since in one revolution the crank pin moves over the complete circumference of the circle $= 2\pi r$, in one second it moves $2\pi rn$ ft. $= 2\pi n \,.\, r = pr = p\mathscr{s}$, since $p = 2\pi n$ and $r = \mathscr{s}$. This must be the speed of the crank pin in feet per second, and it is the same for all positions of the crank. But to find the speed of the piston we must not only know the speed of the crank pin, but the direction in which it is moving, as it is only the vertical component of the motion of the crank pin which affects the piston. Now it is evident that the crank pin is at every instant moving along the circumference of the circle, and the direction of its motion at each instant is therefore along the tangent to the circle, which is at right angles to the radius or crank in every position. We shall then represent the velocity of the crank pin completely by setting out a length which represents the speed of the crank pin $p\mathscr{s}$ to some scale, in a direction perpendicular to the crank. As the crank revolves this line revolves with it, keeping always perpendicular to the crank; and since it is only the vertical component of the motion of the crank pin which is communicated to the piston,

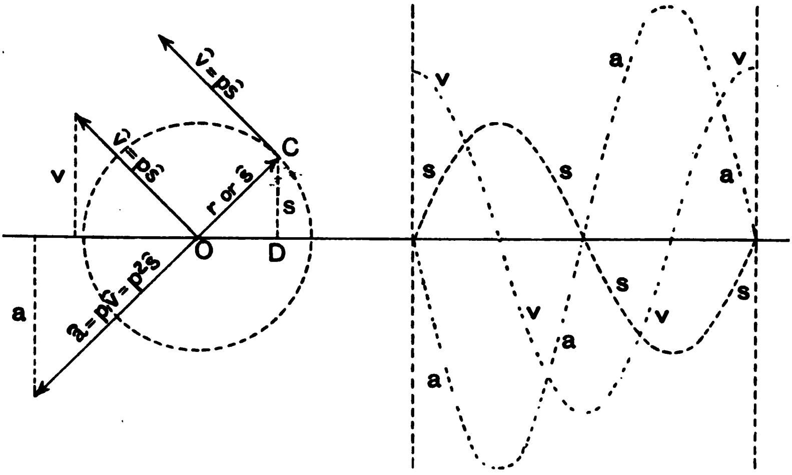

FIG. 47.—Displacement, velocity, and acceleration.

the velocity of the piston is represented by the vertical component of the line representing the crank-pin velocity. This is most conveniently exhibited by setting off the velocity of the crank pin from the centre as a line equal and parallel to the former, *i.e.* of length $p\mathfrak{s}$ and perpendicular to the crank; the vertical height or component v of this line will then represent the piston velocity. As the crank revolves the crank-pin velocity line revolves with it, and the piston velocity v varies from a maximum, when the crank is horizontal or the piston is in the middle of its stroke, to zero when the crank is vertical or the piston is at the end of the stroke, as is obviously true. The maximum velocity of the piston is evidently also represented by the total length of the crank-pin velocity line, or the maximum velocity of the piston is equal to the speed of the crank pin, which is also evident since, when the crank is horizontal, the crank pin is moving up vertically and dragging the piston with it. Hence, if we denote the maximum velocity of the piston by $\mathfrak{v}$, we shall have $\mathfrak{v} = p\mathfrak{s}$, or the maximum velocity of the piston is p times its maximum displacement. We want, however, to know not only the maximum piston velocity, but also its value at any instant. Now it has been seen that the velocity v at any instant is the vertical component of a line of constant length $\mathfrak{v}$ rotating with the same velocity as the crank; or, in other words, v is the vertical component of a uniformly rotating crank $\mathfrak{v}$, and its variation is therefore simple harmonic or sinusoidal, just as the displacement was. The only difference is that the velocity reaches its maximum a quarter of a revolution earlier than the displacement, or it is said to *lead* in front of the displacement by 90°. We hence arrive at the following rule:

If a body move with a simple harmonic displacement of maximum value $\mathfrak{s}$ *and frequency* n *periods per second, its velocity varies in a simple harmonic manner, the maximum velocity* $\mathfrak{v}$ *being* $p = 2\pi n$ *times the maximum displacement* $\mathfrak{s}$ *and leading by a quarter period or* 90° *in front of it.*

Conversely, if a body moves with a simple harmonic velocity of maximum value $\mathfrak{v}$, its displacement is simple harmonic, having a maximum value $\mathfrak{s} = \dfrac{\mathfrak{v}}{p}$, and *lagging* by a quarter period or 90° behind the velocity.

Acceleration.—The remaining quantity that we require to know is the acceleration of the piston. We might set about finding this in a similar way to that by which we found the velocity, *i.e.* by first finding the acceleration of the crank pin, and then taking its vertical component as the acceleration of the piston. But we have an even more simple course open to us now that we have found the velocity. What is the meaning of velocity? It is the rate at which a body changes its position, or, shortly, the rate of change of displacement. Hence what we have just proved is that if a body moves so that its displacement varies in a simple harmonic manner, its velocity or rate of change of displacement also varies in a simple harmonic manner, having a maximum value of p times the maximum displacement, and leading by 90°. This important result may therefore be expressed in the general form:

The rate of change of any simple harmonically varying quantity is itself simple harmonic, and is obtained by multiplying the original quantity by $p = 2\pi n$ and shifting it forwards by a quarter period or 90°.

Or if the original quantity is represented by the vertical component of a crank rotating n times per second, its rate of change is represented by the vertical component of a crank of p times the length of the first crank, leading in front of it by 90° and rotating with it.

Now what is meant by the acceleration of the piston? It is the rate of change of its velocity. But we have just seen that its velocity is a simple harmonically varying quantity. Hence by the above its acceleration will also be simple harmonic, having a maximum value $\hat{a} = p\hat{v}$, and leading in front of the velocity by a quarter period or 90°. Or it is represented by the vertical component a (Fig. 47) of a crank of length $\hat{a} = p\hat{v}$, leading in front of the velocity crank $\hat{v}$ by 90° and rotating with it.

These results can be briefly expressed as follows: *

$$\hat{v} = p\hat{s} \text{ and leads by } 90^\circ \quad \text{or} \quad \hat{v} = p\hat{s} \angle 90^\circ \quad . \quad (11)$$

$$\text{hence} \quad \hat{s} = \frac{\hat{v}}{p} \text{ and lags by } 90^\circ \quad \text{or} \quad \hat{s} = \frac{\hat{v}}{p} \curlyvee 90^\circ \quad . \quad (12)$$

$$\hat{a} = p\hat{v} \text{ and leads by } 90^\circ \quad \text{or} \quad \hat{a} = p\hat{v} \angle 90^\circ \quad . \quad (13)$$

$$\text{hence} \quad \hat{a} = p^2\hat{s} \text{ and leads by } 180^\circ \quad \text{or} \quad \hat{a} = -p^2\hat{s} \quad . \quad (14)$$

since leading by 180° is the same as setting the crank in the directly opposite or negative direction. In fact, as is seen from Fig. 35, the actual acceleration a at any instant bears the same proportion to the displacement s as the maximum acceleration $\hat{a}$ to the maximum displacement $\hat{s}$. Hence we may write $a = -p^2 s$ or the acceleration in simple harmonic motion is always proportional to and opposite to the displacement. This, which is also indicated by the curves showing the variation of s, v, and a at the right of Fig. 47, is the characteristic property of simple harmonic motion, and whenever a body moves in such a way that its acceleration $a = -ks$, where k is a constant,

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* As we shall continually be dealing with lagging and leading quantities, we shall find it convenient to use the symbol $\angle$ for the words 'lagging by,' and the symbol $\curlyvee$ for 'leading by.'

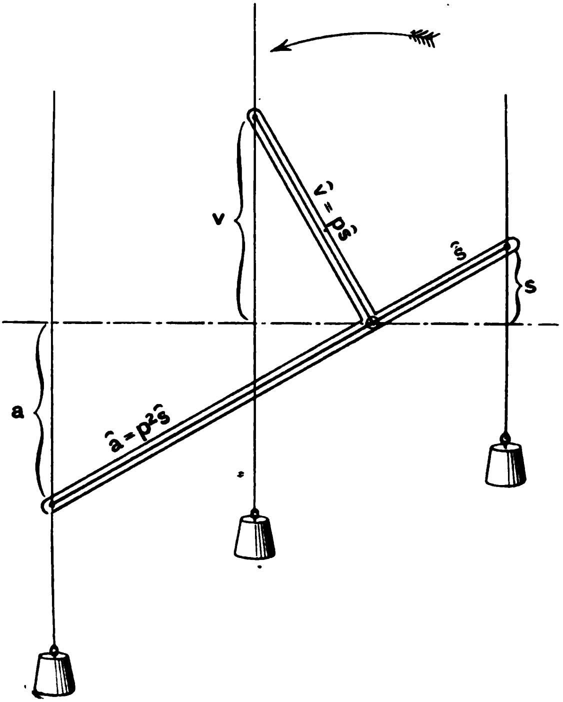

FIG. 48.—Model showing variation of displacement, velocity, and acceleration.

we know that it moves with a simple harmonic motion for which $p^2 = k$ or $p = 2\pi n = \sqrt{k}$ and the frequency $n = \frac{1}{2\pi}\sqrt{k}$. The importance of this will be appreciated later, when we come to deal with resonance and electrical oscillations.

The variation of the displacement, velocity, and acceleration of a piston can be conveniently studied by a model, such as that in Fig. 48, where there are three cranks at right angles, the first being the actual engine crank of length s, the second one which to some scale represents ps, and the third p^2s. If three rods are pivoted at the ends of these cranks and are kept vertical by weights, then as the system of cranks revolves the lengths of the three rods between the end of the crank and the horizontal line through the crank shaft always show the values of the displacement, velocity, and acceleration of the piston at each instant. We can, of course, show the same thing with the engine model.

In order to appreciate the value of these results even for mechanical purposes, let us take the case of a high-speed engine having a length of stroke of 1 ft. and running at a speed of 480 revs. per min., or 8 revs. per sec. Then $\hat{s} = 6'' = {\cdot}5$ ft., $n = 8$, and $p = 2\pi n = 50{\cdot}24$ radians per sec. Consequently the maximum piston speed $\hat{v} = p\hat{s} = 50{\cdot}24 \times {\cdot}5 = 25{\cdot}12$ ft. per sec., and the maximum piston acceleration $\hat{a} = p\hat{v} = 50{\cdot}24 \times 25{\cdot}12 = 1260$ ft. per sec. per sec. Both these quantities are of the greatest importance in the design of the engine. For example, if we suppose that our piston had a diameter of 12″ and an average thickness of 2″, its weight would be about 50 lbs. or, say, 1·5 engineers' units of mass. Hence in order to cause this mass to move backwards and forwards in the cylinder at the above rate, it will require a force $F = ma$ at each instant, or a maximum force at the end of the stroke $\hat{F} = m\hat{a} = 1{\cdot}5 \times 1260 = 1890$ lbs., or nearly a ton.

As another application of these results let us suppose that we have a weight of 2 lbs. hanging from the end of a spiral spring which stretches ·1 ft. for a force of a pound, or has a yield-constant Y of 1 ft. per lb. The mass m of the weight is $\frac{2}{32{\cdot}2} = {\cdot}06$ engineers' units approximately. Now if we pull down the weight a distance s feet, there will be an elastic force due to the spring of $F_1 = \frac{1}{Y}s$ lbs., and when it is released this force will be applied to the weight, giving it an acceleration $a = \frac{F_1}{m} = \frac{1}{mY}s$ ft. per sec. per sec. But when the weight is displaced *below* its normal position the pull of the spring is *upwards*, and hence the acceleration is opposite in direction to the displacement. We may therefore write the acceleration $a = -\frac{1}{mY}s$ where m and Y are obviously constants. But this is of the same form as $a = -p^2s$, consequently the weight will oscillate up and down with a simple harmonic motion for which $p = \frac{1}{\sqrt{mY}}$ or

$n = \frac{1}{2\pi\sqrt{mY}}$ which in our case is $\frac{1}{2\pi\sqrt{\cdot 1 \times \cdot 06}} = 2{\cdot}06$ periods per sec. This experiment can be performed in a few minutes by anyone, by simply putting a known weight on the end of a cylindrical spring, and seeing how far it stretches, thus getting the yield-constant Y. The frequency can then be calculated as above, and checked by a watch, and it will be found that the agreement is usually fairly close, the experimental frequency being somewhat lower than that calculated, owing to the fact that the spring has itself some mass, and that there is a certain small damping effect due to friction, which makes the motion slow down and die away (see Fig. 37, Chap. V). If, however, we take the mass as that of the weight and half the mass of the spring (since one end of the spring is stationary) the agreement is generally very close.

In the case of the boat moored by the rope dealt with on p. 53 we had $Y = \cdot 1$ and $m = 300$, from which $\sqrt{mY} = \sqrt{30} = 5{\cdot}48$, and the time for one complete period $T = \frac{1}{n} = 2\pi\sqrt{mY} = 34{\cdot}4$ secs. If we look at the curves in Fig. 34 we see that this time agrees with the interval between two successive swings in the same direction.

A great deal of help in thoroughly realising these principles is gained by watching a pendulum or by swinging a dumb-bell. In the former case (Fig. 49) it is, of course, evident that the displacement from its normal resting position is greatest at the end of the swing, while the velocity is greatest at its centre. Furthermore, if we consider a swing from left to right, the greatest speed towards the right occurs at the centre of the swing, a quarter of a period before the greatest displacement to the right, showing that the velocity does lead a quarter of a period before the displacement. The same thing also evidently holds good on the back swing. Finally, it is evident that there is no acceleration at the centre of the swing, as the pendulum is at its lowest point, and is consequently moving horizontally so that the earth's pull does not affect it, and, moreover, the acceleration must be zero since the velocity is a maximum. It is evident, also, that the maximum acceleration is at the ends of the swing when the pendulum bob is at its highest point, and the effect of the earth's pull is greatest. Considering the left-hand end of the swing, it is clear that the acceleration is towards the right, since the earth's pull tends to move it to the lowest point, and the speed increases from the end of the swing towards the centre. The maximum acceleration towards the right is therefore at the left-hand end of the swing, and a quarter period earlier than the maximum velocity, as we proved. In fact, to put the whole matter very crudely in a nutshell, we can say that if we wish to reach a certain destination, we must start before we go, and go before we arrive. By starting is here meant the getting up speed or acceleration, and by going the actual velocity or moving towards the goal. There is nothing really more complicated about these differences of phase or of lag and lead than there is in the above simple statement, and, as we shall see later, this is the whole explanation of lag and lead in alternate-current circuits.

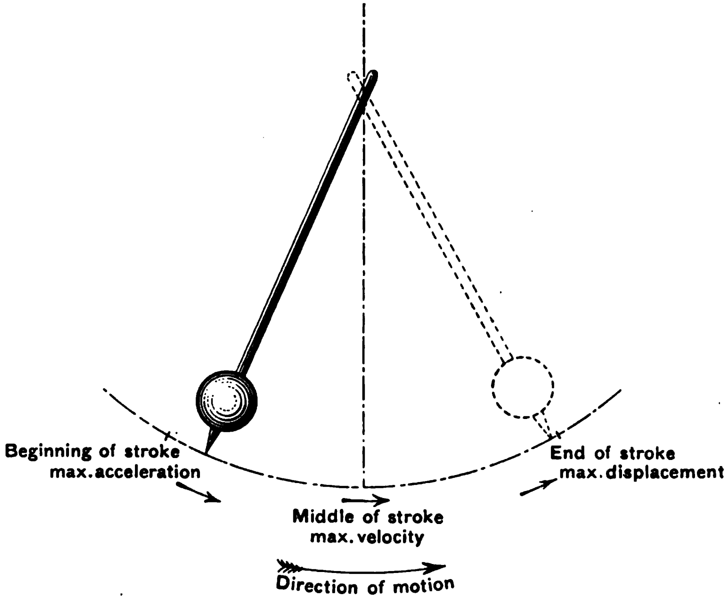

FIG. 49.—Displacement, velocity, and acceleration of pendulum.

When swinging dumb-bells or any heavy weight, the same phenomena can be studied with the additional advantage of our being able to feel the force required to produce the acceleration. The rapid swinging of a heavy iron door is even better, as the weight is taken up by the hinges and we only feel the inertia effects. We shall return to these phenomena later when dealing with alternating currents.

CHAPTER VII

FURTHER PROPERTIES OF SIMPLE HARMONIC MOTION

As simple harmonic motion is the fundamental type of motion with which we deal in alternate-current problems, it must be thoroughly understood; and therefore certain terms employed in conjunction with it will now be defined exactly, and a few more propositions concerning it will be established.

We have already defined simple harmonic motion in the last chapter as the motion of a point along the diameter of a circle which follows the projection on that diameter of a point uniformly rotating round the circle. The mode of variation of this motion is shown in Fig. 46, or in the curves of Fig. 47. We can now explain the following terms.

Period or Cycle.—In one complete rotation of the crank the piston or displacement goes through one complete set of changes and returns to the point from which it started, ready to begin a new set. This complete set of changes is called a *period* or *cycle* of the motion.

Periodic Time.—The time taken for one revolution of the crank or one cycle of the motion is termed the *periodic time.*

Frequency or Periodicity.—The number of rotations of the crank or complete periods of the motion per second is called the *frequency* or *periodicity* of the motion. It is denoted by the symbol $\sim$. For example, the ordinary alternating current for lighting purposes which flows 50 times in one direction and 50 times in the opposite direction per second, or which reverses 100 times per second, would be said to have a frequency of 50 periods per second, or of 50 $\sim$. The number of *alternations* is evidently double the number of *periods.*

Maximum Value or Amplitude.—The greatest value of the displacement or maximum height of the curve is equal to the radius of the crank, and is called the maximum value or *amplitude* of the simple harmonic motion.

The Angular Velocity p of the crank is $2\pi n$ radians per second where n is the frequency in ~ per second.

Phase Angle or Phase.—The angle θ, through which the crank has moved from its central position, is called the phase angle or phase of the motion. It is equal to pt or $2\pi nt$.

Phase Difference — Lag — Lead.—Two simple harmonic motions are said to *differ in phase* by an angle ϕ when there is an angle ϕ between the cranks which would produce these motions (see Fig. 50), and this is called the *angle of phase difference* between them. If the crank B is behind the crank A as they rotate together, the motion or curve B will be later or shifted along the axis of time beyond the curve A, and the simple harmonic motion B is said to *lag* by an angle ϕ behind the motion A. If the crank B is in front of the crank A the curve B will be shifted to the left of curve A, and the motion B is said to *lead* in front of the motion A. When the two cranks are at right angles to one another so that the angle of phase difference is 90°, the motions are often spoken of as being *in quadrature.*

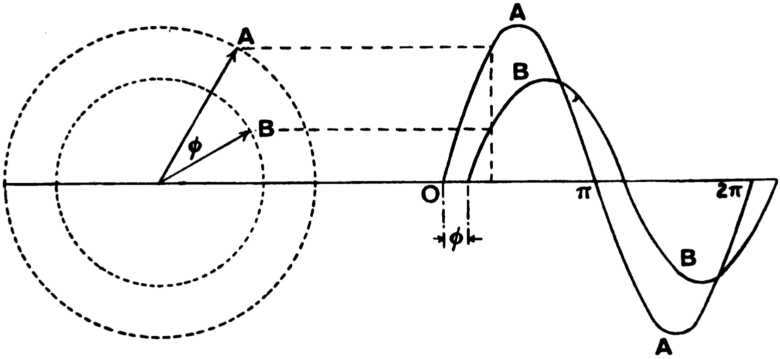

FIG. 50.—Phase difference of two simple harmonic curves.

General Equation.—The essential particulars required to be known about any simple harmonic motion are its amplitude $\mathfrak{s}$, its frequency n, or angular velocity $p = 2\pi n$, and its phase angle a, by which its crank departs from its zero position at the zero of time. The general equation to the motion is evidently $s = \mathfrak{s} \sin(\theta - a)$ or $s = \mathfrak{s} \sin(pt - a)$, as is seen at once from Fig. 51, where the crank, instead of having been horizontal at the zero of time, was inclined at an angle a below the horizontal, and has therefore only reached the inclination $\theta - a$ when it should have reached the angle θ.

We now come to an important matter. From what has been already said, simple harmonic motion is the typical motion of pendulums, engines, alternate currents, &c., and we have learnt how to find the displacement, velocity, and acceleration of such a motion, and hence the forces required to produce it for combinations of elasticity, friction, and inertia. But the next question is: What do we mean by the displacement, velocity, and acceleration or the force when dealing with such motions? They all vary in a simple harmonic manner, being sometimes in one direction and sometimes opposite. Shall we speak of them by their maximum or their average values, or in some other way?

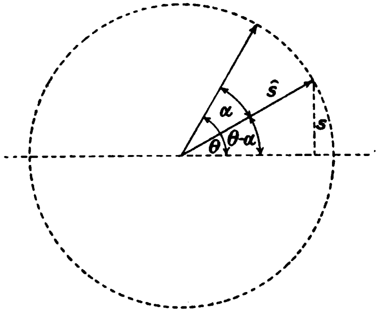

FIG. 51.—General expression for simple harmonic motion.

The simplest way of defining the amount of a simple harmonically varying quantity would, of course, be by taking its maximum value; and, indeed, this is usual in mathematical treatises on such motions. But from our point of view it is hardly satisfactory. If we have a simple harmonic P.D. of a certain amplitude it is evident that it cannot be as effective for lighting, &c., as a steady P.D. of the same value as the amplitude; for, apart from the question of its reversal, the alternating P.D. only attains its maximum for an instant of time, and its average effect is evidently considerably less than the maximum.

Average Value of a Simple Harmonic Motion.—It does not need much reasoning to see that the true average of any simple harmonic motion is zero. The motion is evidently perfectly symmetrical on each side of the central or zero position, and the positive and negative motions exactly balance one another. This can also be easily shown experimentally in the case of an alternating current or P.D. by using a polarised magnetic ammeter or voltmeter, such as a moving-coil instrument. Such an ammeter or voltmeter is influenced by a force or turning moment proportional to the current at each instant, and the indications of the pointer tend to follow the current in either direction. But if the variations in the current about the average value are very rapid, the inertia of the coil prevents it from following these variations, and the pointer simply shows the average value of the current or P.D. Such an instrument when connected to an alternate-current circuit of ordinary frequency simply gives no indication showing that the average value of the current or P.D. is zero.* In fact, as an alternate current is produced by the threading in and out, or reversal of a magnetic flux in a coil, it can easily be shown that the negative and positive quantities must be equal, whatever the wave form. It will at once be apparent, therefore, that it is not the average value of a simple harmonic motion that we ought to measure.

But there is another way of measuring the average which may at first sight appear more rational. An engine piston will do work in whichever direction it is moving, and an electric current will light glow-lamps equally well when reversed. We might therefore proceed to find the average value or height of the curve, taking all the loops as positive, and this we will now do.

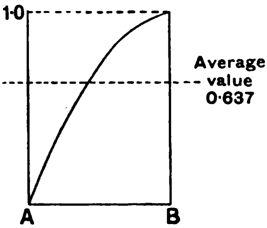

FIG. 52.—Mean height of sine curve.

It is evident again from symmetry that the mean height of the whole curve in this case will be the same as the mean height of the portion from *A* to *B*, Fig. 52, while the crank turns through a right angle, since the rise and fall in the other three quadrants must be precisely similar.

Now, in the time when the crank pin turns from *A* to *B* through a distance $= \frac{\pi}{2} s$ the piston of an engine would move through half the stroke $= s$. Hence the average speed of the piston is to the speed of the crank pin as s is to $\frac{\pi}{2} s$ or as $2 : \pi$. But the maximum speed of the piston, as we saw on p. 82, is equal to the speed of the crank pin, and thus we have the relation $\frac{\text{average speed of piston}}{\text{maximum speed of piston}} = \frac{2}{\pi}$ or the average speed is $\frac{2}{\pi}$ the maximum speed. But we further learnt in the last chapter that if a body moves with a simple harmonic motion its speed or velocity also varies in a simple harmonic manner, and we thus see that the average value of a simple harmonic quantity is $\frac{2}{\pi}$ or ·637 of its maximum value if we consider positive values only.

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* This will usually be found to be the case with good moving-coil instruments. But in instruments where the magnetic effect of the coil is sufficient to affect the strength of the magnet, as in many moving-needle galvanometers, there is a resultant effect tending to produce a steady reading in either direction, as the magnetic needle behaves partly like a soft iron one.

This value can be easily checked by ordinary trigonometrical tables. For we saw in the last chapter, and again at the commencement of the present one, that the variation of a simple harmonic motion is represented by the variation of the sine of an angle as it increases uniformly. In fact, the simplest expression for the displacement of the piston was $s = \hat{s} \sin \theta$, where θ is the angle which the crank makes with the horizontal. If $\hat{s}$ is unity $s = \sin \theta$, and the average value of s over a quarter turn of the crank will be the average value of $\sin \theta$ as θ changes from 0 to 90°. Putting down the values from the tables as follows we have—

θ	5°	15°	25°	35°	45°	55°	65°	75°	85°
$\sin \theta$	·0872	·2588	·4226	·5736	·7071	·8192	·9063	·9659	·9962

and adding these we get 5·7369, which, divided by the number of values, *i.e.* 9, gives us the mean value as ·6341. This is nearly equal to the value ·637 found above, and would have been exactly equal to it if more values of the sine had been taken.

Effective Value of a Simple Harmonic Quantity.—But although we have now found the average height of the curve, we have no right to assume that it represents the utility of the reciprocating motion or alternating current as compared with a steady motion or current. In order to ascertain what value we ought to take we must consider the actual effect of the motion or current. Now the main object of motion or electrical supply is to give out energy or to do work. In electric lighting, for example, it is the average supply of energy to the lamps that decides their brightness, and it is the average horse-power given out by an engine during its revolution that concerns us. Now a motion can be employed to compress a spring, to move a body against friction, or to overcome inertia, and an electric current may have to flow through a condenser, a resistance or an inductance. Taking the mechanical cases first, we know that when we compress a spring, the force $F_1 = \frac{1}{Y}s$ is proportional to the displacement. We also know that if a body is displaced from s_1 to s_2

under the action of a force F in the direction of the displacement, the work done in ft.-lbs. is the force in lbs. multiplied by the displacement in feet or $F(s_2 - s_1)$. But if we plot a curve connecting s and F (Fig. 53) for the spring of Fig. 17, we see that if $s_2 - s_1$ is small, $F(s_2 - s_1)$ is the area of the small shaded strip, and therefore the whole work e done in compressing the spring by an amount s is the area of the triangle OFS, which is half the base multiplied by the vertical height, or $\frac{1}{2}Fs$. Since $F = \frac{1}{Y}s$, $e = \frac{1}{2}\frac{1}{Y}s \times s = \frac{s^2}{2Y}$, or is proportional to the *square* of the displacement s. We have also, since $s = YF$, $e = \frac{1}{2}Fs = \frac{1}{2}YF^2$, or the work is proportional to the *square* of the force. If, on the other hand, the force is employed in overcoming friction, we have $F = fv$, and we saw in Chap. IV that the power was the product of the force and velocity, or $P = Fv = fv \times v = fv^2 = \frac{F^2}{f}$, which is again proportional to the *square* of the velocity or to the *square* of the force. Finally, as we saw in Chap. IV, the kinetic energy in a moving mass is $\frac{1}{2}mv^2$, which is again proportional to the *square* of the velocity.

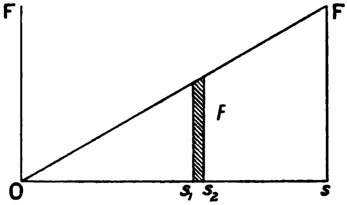

FIG. 53.—Work done by spring.

Turning to electrical illustrations, the energy stored in a condenser is $\frac{Q^2}{2K}$ or $\frac{1}{2}KV^2$ as shown on p. 42; the power absorbed by a resistance rC^2 or $\frac{V^2}{r}$, and the energy imparted to an inductance $= \frac{1}{2}LC^2$, in each case proportional to the square of the quantity, current, or P.D. respectively.

It is clear, therefore, that the average energy or power is proportional to the average or mean *square* of the force or P.D., &c., instead of to its ordinary mean or average value. This at once removes the difficulty of reversal of direction, or of positive and negative signs, as the square of either a positive or a negative quantity is always positive, and the average or mean value of the square of a reciprocating quantity is therefore not zero.

We can now see exactly what is the most convenient value to be taken as the effective value of a simple harmonic motion. It will be most convenient to consider the case of an electric current, but the same reasoning would equally apply in the other cases.

Let us define the effective value of a varying current as the value of a steady current which supplied the same average amount of power to a resistance or glow-lamp, and thus develops the same average amount of heat and light. Then we know that the power at each instant $w = rC^2$. Hence the mean power $\bar{w} = \text{mean } rC^2 = r \times \text{mean } C^2$, or the average power in watts is r times the mean or average value of the square of the current. For since r is supposed to be constant, it makes no difference in calculating whether we take the average value of rC^2 or the average value of C^2 and multiply the result by r afterwards.

Now suppose that we pass a steady current of strength $\bar{C}$ amperes through the same resistance. We shall then have a steady supply of power $= r\bar{C}^2$, and we can regulate the value of the current $\bar{C}$ until the resistance is as hot as it was with the varying current. Then this means that the steady power consumption with the steady current must be equal to the average power consumption with the unsteady current, and hence $r\bar{C}^2 = r$ mean C^2. Cancelling r on the two sides of the equation, we have—

$$\bar{C}^2 = \text{mean } C^2$$

and consequently

$$\bar{C} = \sqrt{\text{mean } C^2} \quad . \quad . \quad . \quad . \quad (15)$$

In other words, *a steady current produces the same heating effect as a varying one when the value of the steady current is the square root of the average square of the varying current. This steady current is called the effective or virtual or R.M.S. (Root Mean Square) value of the varying current.*

It is clear, therefore, that if we have an alternating current and wish to know its effective value, *i.e.* the magnitude of the direct current which would produce the same heating effect, we must find the mean square of the value of the alternating current, and take the square root of this result. Since also the power $rC^2 = \dfrac{V^2}{r}$ we can prove in like manner that the effective value of an alternating P.D. is its $\sqrt{\text{mean square.}}$

Although it is not intended in the present volume to deal with practical testing, a few words may profitably be devoted to methods of measuring alternating currents and differences of potential, as the writer has generally found that the reason for using the square root of mean square value and its meaning is somewhat of a stumbling-block to students.

If the wave of the current is known, its effective value can of course be obtained by calculation. Fig. 54 shows an ondograph curve of an alternating current C^2 of distorted wave form. The heights of this curve have been scaled off for angles of 10°, 30°, 50°, &c., and entered in the following table as in taking the heights of the strips in an indicator diagram, and the mean height of the half-wave is found to be exactly 1 ampere. We now enter the squares of these heights in the third column, and these, when plotted, give us the curve marked C^2, which does not reverse sign. The mean height of this curve or mean square of current is calculated to be 1·271, and is shown by the upper dotted line in the diagram, and the square root of this value is 1·128 amperes, which is the effective value of the alternating current, and is represented by the thick dotted line. It will be noticed that this value is considerably higher than the ordinary mean value. The ratio of the effective to the ordinary mean value of the curve is called the *form factor* of the wave, and in this case is $\frac{1·128}{1·00} = 1·128$.

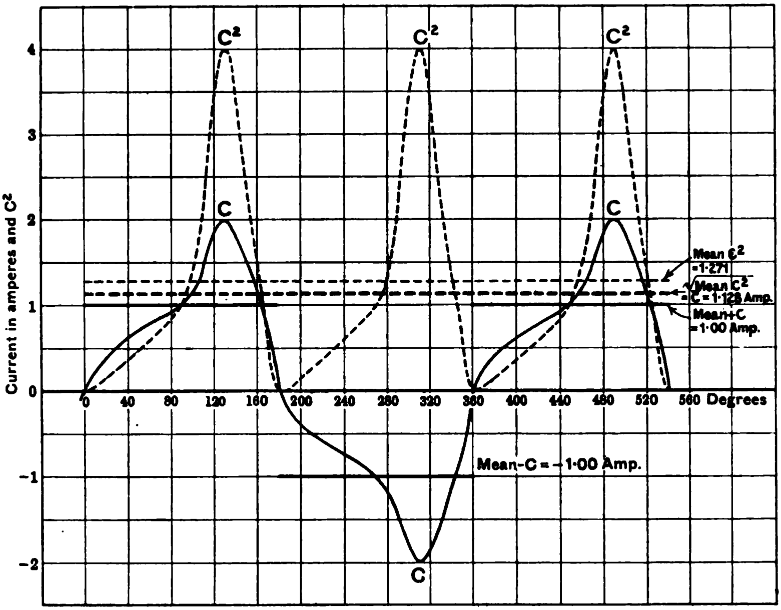

FIG. 54.—Curves of C^2, mean C^2 and $\bar{C}$.

TABLE IX.

Angle in Degrees.	Current in Amperes C.	C^2.
10	·25	·0625
30	·50	·25
50	·70	·49
70	·85	·723
90	1·00	1·00
110	1·45	2·105
130	2·00	4·00
150	1·50	2·25
170	·75	·5625
Sum	9·00	11·443
Mean	1·00	1·271
$\sqrt{\text{Mean square}}$	—	1·128

We have seen that the real criterion of the value of an alternating current is the heating which it produces in a resistance, and that the effective value of the alternating current is the value of the direct current which produces the same heating in the same resistance. This is the basis of a number of the methods of measuring alternate currents. If we were to mount a glow-lamp on a photometer bench (Fig. 55) and pass the alternating current to be measured through it, we could balance it against a steady source of light, and then change it over to a direct current, which can be varied until the photometric balance is restored. We then know that the light emitted by the lamp and hence the heating of the filament is the same with the direct and with the alternating current; the reading of an ammeter in the direct-current circuit therefore gives the effective value of the alternating current through the lamp. This method has actually been employed by Kurlbaum in the Berlin Reichsanstalt, except that in order to measure large currents he employs platinum tubes instead of glow-lamp filaments, and makes use of a special form of photometer. This method has been described first in order to draw attention to the equality of the light produced by a direct current and by an alternating current of the same effective value, but as long as we have a conductor raised to the same temperature by the direct and alternating currents it will serve. There are two convenient methods of knowing the temperature: by the expansion of the conductor, and by a thermo-junction. The hot-wire ammeters and voltmeters are well known, and when calibrated by direct currents must, therefore, give directly the effective values of alternating currents or voltages.

Any slowly moving instrument in which the force or torque is proportional to the square of the current or P.D. when calibrated on direct-current circuits, will give effective values directly on alternate-current circuits. For if the torque is proportional to the square of the current or P.D., the mean torque is proportional to the mean square, and the reading to the $\sqrt{\text{mean square}}$ value. The attraction of a magnet or solenoid for an unsaturated iron core is proportional to the square of the current, since it is proportional to the strength of the current in the coil and the magnetisation of the iron, which is itself proportional to the current at low inductions. The repulsion of two pieces of iron in the same coil comes under the same category, since each of them is magnetised proportionally to the current, and the repulsion between them is proportional to the product of their magnetic strengths, and hence to the square of the current. Similarly, the force or torque between two coils in series carrying the current, as in the Kelvin balance or Siemens dynamometer, is proportional to the square of the current.

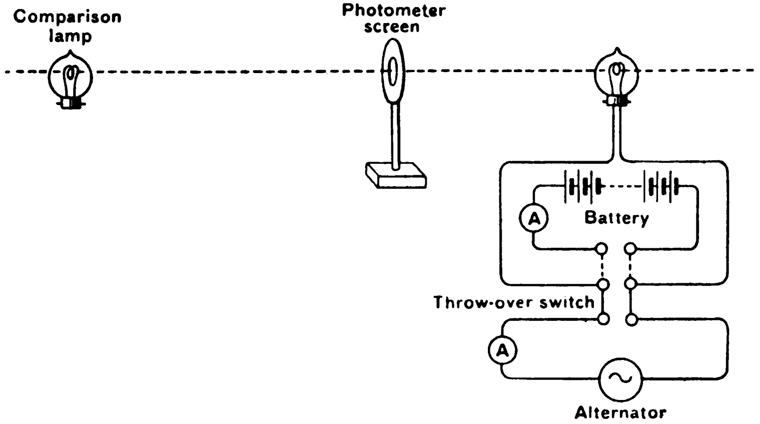

FIG. 55.—Lamp photometered with direct or alternate current.

Finally, the attractive force between two plates is proportional to the square of the P.D. between them, and consequently the force or torque in an electrostatic voltmeter is proportional to the square of the P.D. All these instruments—in fact, practically all instruments which do not contain permanent magnets—obey a square law, and therefore naturally measure the effective value of an alternate current.*

It will therefore be seen that, apart from the $\sqrt{\text{mean square}}$ value of a current representing the same power, heating, or lighting as a direct current of the same numerical value, it is the value which is naturally given by practically all instruments which will give any reading with alternate currents. There can be no hesitation, therefore, in accepting the $\sqrt{\text{mean square}}$ value as the most suitable.

Effective Value of a Simple Harmonic Motion.—Instead of the mean or average value of the half-wave of a simple harmonic curve, which we found on p. 91, what we want to know is the effective or $\sqrt{\text{mean square}}$ value of such a curve. As in the case of the mean value there are several ways of finding this value, but probably the simplest and most satisfactory method is to turn to our engine model.

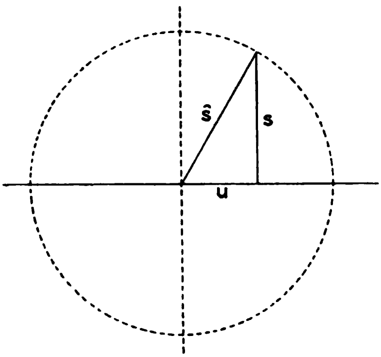

FIG. 56.—Displacement of horizontal piston.

If $\hat{s}$ represents the uniformly rotating crank then the vertical component s is the displacement of a vertical piston from its central position, and we know that as the crank rotates the piston moves with a simple harmonic motion. We therefore want to know the $\sqrt{\text{mean } s^2}$ during the rotation, and this will be the effective value of s, or $\bar{s}$. Now suppose that the crank pin, as well as being connected to a piston in a vertical cylinder, is also connected to a second piston in a horizontal cylinder. It is evident that just as the displacement of the vertical piston was given by the vertical component of the crank or s, so the displacement of the horizontal piston from its central position will be given by the horizontal component of the crank, which is denoted by u in the above diagram (Fig. 56). It is evident also that as the crank rotates, the displacement u varies in a simple harmonic manner just like s, with the same maximum value or amplitude and the same frequency. The only difference between the two motions, in fact, will be in their phases, which will be in quadrature, but this evidently does not affect their maximum or average values.

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* Alternate-current instruments when calibrated on direct-current circuits do not always read quite correctly owing to the presence of eddy currents or of inductance in the coils. But these effects are in the nature of errors from the fundamental principle, and can generally be eliminated by good design of the instrument.

Hence we may agree that the average or mean value of u^2 is the same as that of s^2.

But s, u, and $\hat{s}$ form a right-angled triangle in every position of the crank, and hence $s^2 + u^2$ always $= \hat{s}^2$. Consequently—

$$\text{mean } s^2 + \text{mean } u^2 = \text{mean } \hat{s}^2 = \hat{s}^2$$

as $\hat{s}$ remains constant all the time.

But we saw above that mean $s^2 =$ mean u^2. Therefore—

$$\text{mean } s^2 + \text{mean } u^2 = 2 \times \text{mean } s^2 = \hat{s}^2, \text{ or mean } s^2 = \tfrac{1}{2}\hat{s}^2$$

Thus, when we have a simple harmonic motion, the mean square of the displacement is half the square of its maximum value. Taking the square root of both sides we have—

$$\sqrt{\text{mean } s^2} = \bar{s} = \sqrt{\tfrac{1}{2}\hat{s}^2} = \frac{1}{\sqrt{2}}\hat{s} = \frac{1}{1{\cdot}414}\hat{s} = {\cdot}707\hat{s} \quad . \quad (16)$$

i.e. the effective value of a simple harmonic motion is got by dividing its maximum value by $\sqrt{2} = 1{\cdot}414$, or is 70·7% of its maximum value. Conversely, if we know the effective value of a S.H.M. and wish to know its maximum value, we have—

$$\hat{s} = \sqrt{2} \, . \, \bar{s} = 1{\cdot}414\bar{s} \quad . \quad . \quad . \quad . \quad (17)$$

This result is so important that it is well for us to establish it independently. We can, of course, prove it from trigonometrical tables as we did when finding the ordinary mean value.

In the accompanying table we have put down the values of $\sin^2 \theta$ for angles of 5, 15, 25, &c., degrees as follows:

θ	5°	15°	25°	35°	45°	55°	65°	75°	85°
$\sin \theta$	·0872	·2588	·4226	·5736	·7071	·8192	·9063	·9659	·9962
$\sin^2 \theta$	·0076	·0670	·1785	·3290	·5000	·6710	·8215	·9330	·9924

The sum of these values of $\sin^2 \theta = 4{\cdot}500$, which, divided by the number of them $= 9$, gives the mean value as ·500 or $\tfrac{1}{2}$. Hence the mean value of $\sin^2 \theta = \tfrac{1}{2}$ and its maximum value is 1; the mean square is therefore half the maximum square or $\sqrt{\text{mean square}} = \frac{1}{\sqrt{2}}$ times the maximum, as we found above.

We could find the same thing with a little trigonometrical working, for $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, and since the mean value of $\cos 2\theta$ is 0 the mean value of $\sin^2\theta = \frac{1}{2}$.

It may therefore be taken as established that the effective value of a simple harmonic motion, P.D., or current is $\frac{1}{\sqrt{2}}$ or ·707 times its maximum value, and the importance of this will appear shortly. The ordinary mean value of the half-wave was $\frac{2}{\pi}$ or ·637 times the maximum. Hence the form factor of a simple harmonic motion $= \frac{\sqrt{\text{mean square}}}{\text{mean}} = \frac{1}{\sqrt{2}} \div \frac{2}{\pi} = \frac{\pi}{2\sqrt{2}} = 1{\cdot}11$, a quantity which it is sometimes useful to know as it indicates how peaked the wave form is.

A pretty experiment, by which the fact that the maximum value of an alternate-current P.D. is about 1·41 times its effective value, can be performed by anyone who has access to an ordinary electrostatic voltmeter on an A.C. circuit. If such an instrument indicates 100 volts, *i.e.* that the effective value of the alternating P.D. is 100 volts, then if we detach one of the wires from the terminals, or break the voltmeter switch, we shall sometimes find that the deflection falls away, but sometimes that it actually goes up, and the deflection will occasionally be as high as 141 volts (or even more if the wave form is more peaked than a sine curve). The explanation of this is that an electrostatic voltmeter is like a condenser, and if it is charged and disconnected it will remain charged, and the reading will be that of the voltage at the moment of disconnection. When the connection is broken the voltage may have any value between zero and the maximum value, so that the deflection may go down or up, but after a number of trials the disconnection may happen to take place at the instant when the P.D. is at its maximum, and the needle will deflect to this value. It need hardly be said that it is essential that the insulation of the instrument should be very perfect for this to take place, otherwise the effect will be entirely spoilt by leakage, but the writer has frequently seen it on switchboard instruments of the Kelvin type in hot engine-rooms, and it can nearly always be seen by shunting the voltmeter with a condenser of about a microfarad capacity, which helps to hold the charge.

CHAPTER VIII

COMPOSITION AND RESOLUTION OF SIMPLE HARMONIC MOTIONS

WE now come to one of the most essential parts of our subject. In Chapters I, II and III we learnt how to find the forces required to move a body against its elasticity, friction, or inertia, and in Chapter V we considered how we should find the force or motion when these properties existed together, as was usually the case. Since then we have learnt how to find the displacement, velocity, and acceleration for a simple harmonic motion, and hence how to find the force required to produce this kind of motion in the case of a spring, friction, or mass. It is evident, therefore, that our next step ought to be to ascertain the force required to move a body in a simple harmonic manner when it combines the three properties, elasticity, friction, and inertia; by doing this we ought at once to be able to find the voltage to produce a simple harmonic electrical motion or alternate current in a circuit having capacity, resistance, and inductance.

Now it has been already shown that when a body moves in a simple harmonic manner its velocity and acceleration are also simple harmonic, and hence the force required to impart this motion to either a spring, friction, or inertia must be simple harmonic. In order, therefore, to get the total force for a combination of these properties we must add together the forces required to stretch the spring, to move against the friction, and to accelerate the mass. In other words, we have to get the sum of three simple harmonically varying forces, at every instant, and thus arrive at the variation of the total force.

For example, let us take our engine running at 480 revs. per min. with a stroke of 1 ft. or maximum displacement of 6″, and suppose it coupled directly to a second piston moving in a closed cylinder like an air-cushion, or to a spring. Then if we take a yield-constant of ·001 ft. per lb., a friction-constant of

50 lbs. for a velocity of 1 ft. per sec., and a total mass of 64·4 lbs. or 2 engineers' units, we have, as before, $p = 2\pi \times 8 = 50{\cdot}24$, $\hat{v} = p\hat{s} \angle 90^\circ = 25{\cdot}12 \angle 90^\circ$, and $\hat{a} = -p^2\hat{s} = -1260$ ft. per sec. per sec. Hence $\hat{F}_1 = \frac{1}{Y}\hat{s} = \frac{1}{\cdot 001} \times \cdot 5 = 500$ lbs., $\hat{F}_2 = fv = 50 \times 25{\cdot}12 = 1256 \angle 90^\circ$ lbs., and $\hat{F}_3 = ma = -2 \times 1260 = -2520$ lbs. The curves representing these three forces are shown dotted in Fig. 57.

If we now want to find the total force required we have simply to take $F = F_1 + F_2 + F_3$ or to add the ordinates of the three curves together, and we get the thick black curve in the figure which rises and falls in a somewhat similar way to each of the separate curves from which it was obtained. But what sort of curve is this, and how can the force be calculated without going through the process of drawing the separate curves and adding them together? Since all the three dotted curves are simple harmonic, what we have really to do is to find the result of adding simple harmonic curves or motions together, and the simplest method of doing so. This is known as the composition of simple harmonic motions.

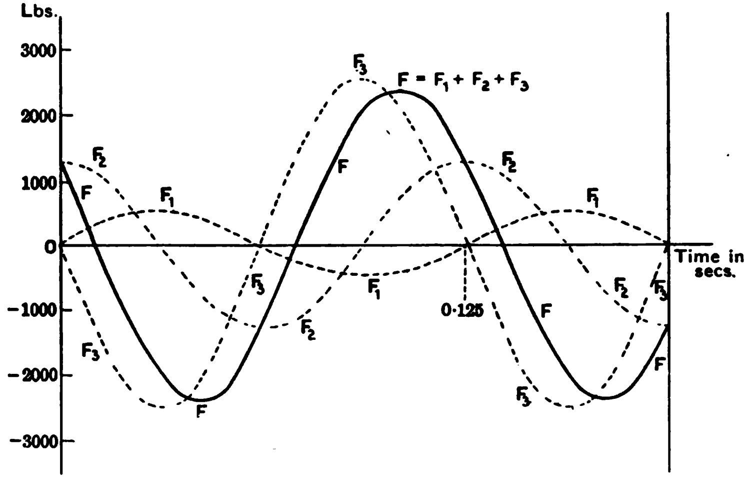

FIG. 57.—Combination of forces for elasticity, friction, and inertia.

Let us suppose that we have two simple harmonic curves, marked *A* and *B* on the right of Fig. 58, of any amplitude and phase but having the same periodic time or frequency. Then we know that if we draw two circles of radii equal to the maximum heights of curve *A* and curve *B* respectively, and project the vertical heights of the curves at a certain time on these circles, we obtain two cranks or vectors *OA* and *OB*, which if rotated with a speed equal to the frequency would impart a motion to a vertically moving piston exactly corresponding to the original curves. We now want to find the result of adding these two curves together, *i.e.* of adding their vertical heights together at each interval of time. But we have learnt from our engine that the vertical height of the curve is the same as the vertical height of its crank pin above the horizontal. The adding together of the heights of the curves *A* and *B* at every instant is therefore evidently the same as the adding together of the vertical heights *Aa* and *Bb* of the crank pins above the horizontal, for corresponding positions of the cranks.

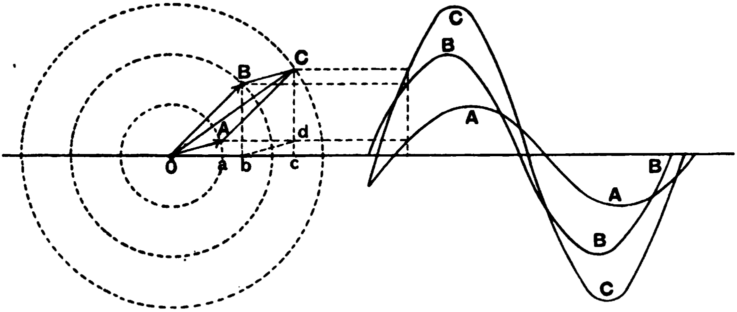

FIG. 58.—Composition of simple harmonic motions.

Now let us take the second crank *OB* and put it on the end of *OA* parallel to itself as at *AC*. Then, since the cranks *OA* and *OB* always rotate together and keep the same angle with one another, owing to their moving with the same frequency, the figure *OACB* is always of the same shape, and the point *C* will move round a circle uniformly at the same rate as the points *A* and *B*. In fact, the point *C* can be considered as being on the end of a crank or vector *OC*, which uniformly rotates with the same frequency. If this crank were attached to the piston it would give it a simple harmonic motion, represented by the upper curve *C* on the right of the figure. But the height of this curve at every instant is equal to the height of the crank pin *C* above the horizontal, or to *Cc*, and this is made up of *dc*, which is equal to *Aa*, and to *Cd*, which is equal to *Bb*. It will be readily seen that this must be true whatever may be the position of the cranks, and therefore the curve *C*, which represents the motion of a piston actuated by the crank *OC*, is exactly the same as that which is obtained by adding the two simple harmonic curves *A* and *B* together. But since *C* is a curve representing the motion of a piston attached to a uniformly rotating crank, it must be simple harmonic and of the same frequency as the original motions. Furthermore, the crank *OC* is obtained by putting the crank *OB* on the end of the crank *OA* and completing the triangle or parallelogram precisely as we find the resultant of two forces or velocities or any vector quantities. Hence we have the following all-important rule:

The resultant or sum of two simple harmonic motions in the same direction and of the same frequency is a simple harmonic motion of the same frequency, the crank or vector of which is obtained by adding together the cranks or vectors of the two original motions as if they were forces or vectors.

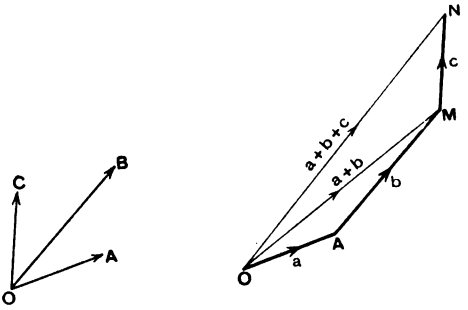

FIG. 59.—Vector composition of simple harmonic motions.

It is obvious, however, that this result is at once capable of extension to any number of such curves or motions. For since the resultant of two such motions represented by their cranks *OA* and *OB* is a simple harmonic motion represented by the crank or vector *OM*, then if we have a third harmonic motion of the same frequency represented by the vector *OC*, and add it to the other two, it is the same as adding it to the simple harmonic curve represented by the vector *OM* (Fig. 59). The result must therefore be a simple harmonic motion represented by the vector *ON*, got by putting *OC* on the end of *OM*, or on the end of *OA* and *OB*, precisely as in adding a number of forces together, and this may be extended to any number. Hence we may generalise our result as follows:

The resultant or sum of any number of simple harmonic motions in the same direction and of the same frequency is a simple harmonic motion of the same frequency, the crank or vector of which is obtained by adding together the cranks or vectors of *all the original motions as if they were forces or vectors, i.e. according to the vector polygon law.*

It will be noticed that in these statements the separate or component motions are said to be in the same direction, *i.e.* along the same line, not in different directions as in the case of horizontal and vertical cylinders, &c. This is of importance to note, as we shall have occasion later, in connection with polyphase currents and rotating fields, to consider what happens when the motions are in different directions.

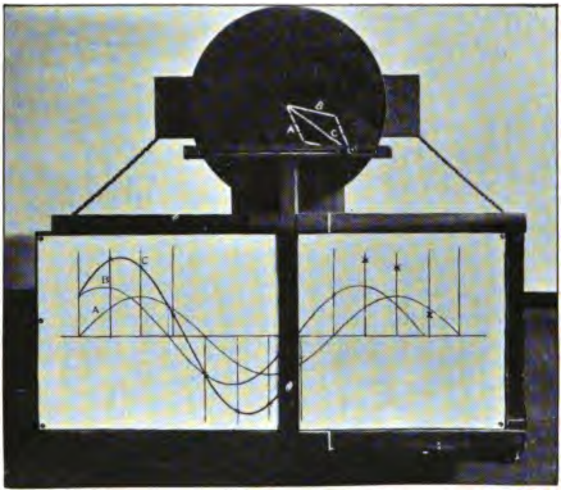

FIG. 60.—Composition of simple harmonic motions by engine model.

The importance of this result cannot be over-estimated, as it means that we can find the resultant of these harmonic motions or the forces or P.D.'s required in combinations of elasticity, friction, and inertia; or of capacity, resistance, and inductance, without drawing the component curves. We will proceed to take an example of this shortly, but in order to clinch the matter it may be illustrated once more by the aid of the skeleton model engine described in Chapter VI. This model, as before mentioned, has a disc or flywheel at any part of the surface of which the crank pin can be fixed, and this surface is divided up into squares for convenience of measurement. We then mark off with chalk a crank on this disc, such as *A* (Fig. 60), and putting the crank pin at the end, the pencil traces out the simple harmonic curve *A* on the paper beneath.

If we now mark off another crank *B* and repeat the process we get the curve *B*. By means of dividers the heights of curve *A* and curve *B* are easily added together, and we get the series of points marked by crosses on the paper below. But we can immediately add the cranks together like forces by the aid of the squares on the disc, obtaining the resultant *C*. On fixing the crank pin at *C* and turning the disc the pencil traces out the curve *C*, which passes exactly through the crosses previously obtained by addition, and which instantly shows the correctness of the method of adding the cranks.

It is now immediately possible to calculate the total force in the case of the engine problem with which we started. We came to the conclusion that the three forces required to overcome the spring, friction, and inertia respectively were $\hat{F}_1 = 500$ lbs. in phase with s, $\hat{F}_2 = 1256$ lbs. leading 90° in front of s, and $\hat{F}_3 = 2520$ lbs. leading 180° in front of s or in the opposite direction to s. Hence these three forces can be represented as in Fig. 61. If we imagine the crank or vector of $\hat{s}$ as being horizontal and to the right, then adopting a scale of 1″ = 1000 lbs., $\hat{F}_1$ will be represented by a line 500 lbs. = ·5″ long in phase with, or in the same direction as, $\hat{s}$. $\hat{F}_2$ will be represented by a line 1256 lbs. or 1·256″ long set vertically upwards or leading 90° in front of $\hat{s}$, while $\hat{F}_3$ will appear as a line 2500 lbs. or 2·5″ long leading 180° or set backwards horizontally. To add these together we have only to do as in Fig. 62, viz. to put the vector $\hat{F}_2$ on the end of $\hat{F}_1$ and the vector $\hat{F}_3$ on the end of $\hat{F}_2$; the resultant $\hat{F}$ will be the crank or vector from the commencement of $\hat{F}_1$ to the end of $\hat{F}_3$. The curve of total force will then be that having this resultant $\hat{F}$ as its crank or vector, and the maximum height of this curve or maximum force will be given by the length of the crank, which

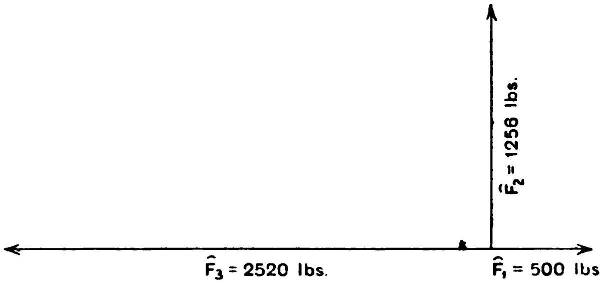

FIG. 61.—Forces for simple harmonic motion.

can be scaled off from the diagram as 2·38 inches corresponding to 2380 lbs. We can also measure the angle ϕ between $\hat{F}$ and $\hat{F}_1$, which is in phase with $\hat{s}$, with a protractor, and it comes out as 148·2 degrees, which is the angle by which the total force $\hat{F}$ leads in front of the displacement. The answer to our problem as to the force exerted by an engine running at 500 revs. per min. with a 12″ stroke, when acting on a spring with a yield-constant of ·001 ft. per lb., a friction-constant of 50 lbs. per ft. per sec., and a mass of 64·4 lbs. is 2380 lbs. $\angle$ 148·2°. This result can be compared with that obtained by adding together the curves in Fig. 57, and is seen to agree exactly.

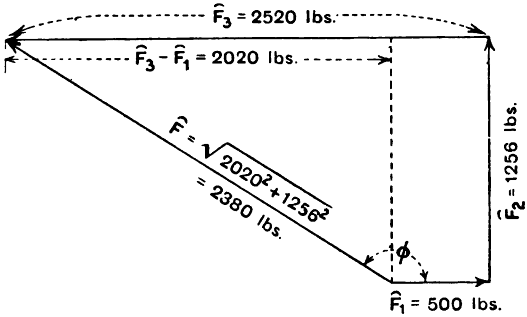

FIG. 62.—Resultant force for simple harmonic motion.

But it is, of course, unnecessary to draw the vectors to scale, as the result can be easily obtained by calculation. The well-known proposition of Euclid (I, 47) tells us that the square on the hypothenuse or side opposite the right angle in a right-angled triangle is equal to the sum of the squares on the other two sides.

In Fig. 62 it is readily seen that $\hat{F}$ is the hypothenuse of a right-angled triangle, of which $\hat{F}_2 = 1256$ and $\hat{F}_3 - \hat{F}_1 = 2020$ are the other two sides. Hence by this proposition $\hat{F}^2 = \hat{F}_2^2 + (\hat{F}_3 - \hat{F}_1)^2$ and $\hat{F} = \sqrt{\hat{F}_2^2 + (\hat{F}_3 - \hat{F}_1)^2} = \sqrt{1256^2 + 2020^2}$ $= 2380$ lbs. as before. At the same time the angle of lead can be determined from the simple relation—

$$\tan\phi = -\frac{\hat{F}_2}{\hat{F}_3 - \hat{F}_1} = -\frac{1256}{2020} = -\cdot 622$$

from which $\phi = 148{\cdot}2°$.

It will thus be seen that the calculation of the force required to impart a simple harmonic motion to a combination of a spring, friction, and inertia is extremely simple. All we have to do is to calculate the three simple forces $\hat{F}_1 = \frac{1}{Y}\hat{s}$, $\hat{F}_2 = f\hat{v} = fp\hat{s} \angle 90°$, and $\hat{F}_3 = m\hat{a} = -mp^2\hat{s}$, and find $\hat{F} = \sqrt{\hat{F}_2^2 + (\hat{F}_3 - \hat{F}_1)^2}$ and $\tan \phi = -\frac{\hat{F}_2}{\hat{F}_3 - \hat{F}_1}$, or scale them off from a diagram.

Resolution of Simple Harmonic Motions.—In dealing with forces it is very often convenient to be able to split them up or resolve them into components as well as to be able

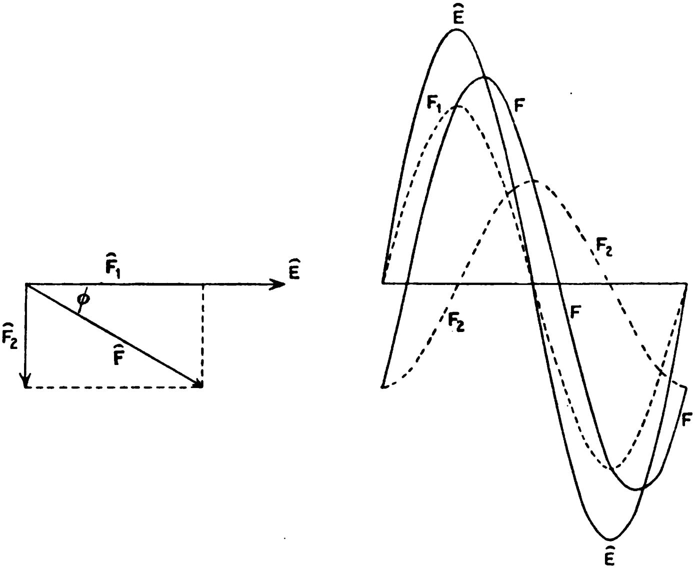

FIG. 63.—Resolution of simple harmonic motions.

to add them together; and we shall see, when dealing with alternate-current circuits in parallel and with alternate-current power, that it is desirable to do the same thing with simple harmonic motions. In Fig. 63 we have a crank or vector $\hat{E}$ which is taken as the standard for reference, and a second vector $\hat{F}$ lagging behind $\hat{E}$ by a phase angle ϕ. Then if we drop perpendiculars from the end of $\hat{F}$ upon $\hat{E}$ and upon a line at right angles to $\hat{E}$ we get two components $\hat{F}_1$ and $\hat{F}_2$, and it is evident that if $\hat{F}_2$ is placed on the end of $\hat{F}_1$ the resultant is $\hat{F}$. The vector $\hat{F}$ is therefore the sum of the two, $\hat{F}_1$ and $\hat{F}$, one of which is in phase with $\hat{E}$, and the other of which lags 90°

behind $\hat{E}$, or is in quadrature with it. The curves on the right of the figure represent the simple harmonic motions produced by the rotations of the cranks $\hat{E}$ and $\hat{F}$ respectively, and the dotted curves show the harmonic motions due to the two cranks $\hat{F}_1$ and $\hat{F}_2$ respectively. The amplitudes of these two motions are equal to $\hat{F}_1$ and $\hat{F}_2$, and it is evident from the diagram that the component in phase $\hat{F}_1 = \hat{F} \cos \phi$ and the component in quadrature $\hat{F}_2 = \hat{F} \sin \phi$.

Hence any simple harmonic motion can be resolved into two components, one in phase with, and the other in quadrature with, another simple harmonic motion. These two components are obtained by multiplying the original motion by the cosine and by the sine of the angle of phase difference respectively.

PART III

ALTERNATE CURRENTS IN CONNECTED CIRCUITS

CHAPTER IX

ALTERNATE CURRENTS

We have now learnt sufficient of the properties of simple harmonic motion and of the forces required to produce such motions, and can therefore proceed to apply our conclusions to alternate currents. First of all, however, we must ask the question: Does a simple harmonic motion represent the variation of an alternate current?

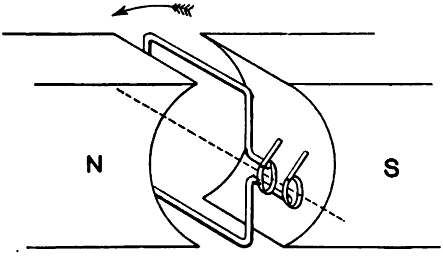

Fig. 64.—Elementary alternator.

To settle this question consider the simplest possible means of obtaining an alternating current.

When a coil of wire rotates between the poles of a magnet, as in Fig. 64, and is provided with slip rings, by which the current can be collected, we know that an alternating E.M.F. is produced. In order to see how this E.M.F. varies, suppose, as in Fig. 65, that the magnetic field is horizontal and uniform, and that the coil rotates uniformly. Now we know that the E.M.F. produced in a conductor at any instant is proportional to its rate of cutting lines of force. Since the field is horizontal, it is only the vertical motion of the conductors which produces any cutting, and the rate of cutting lines is therefore proportional to the vertical component of the velocity of the conductors. But if the coil rotates uniformly each conductor moves round a circle, like the crank of the engine; and the vertical velocity of the conductors, like the velocity of the engine piston on p. 82, varies in a simple harmonic manner. Hence the induced E.M.F. wave must also be simple harmonic.

We can also see without any difficulty how much this E.M.F. must be. For the E.M.F. in one conductor when the coil is in the horizontal plane $= \frac{lB\hat{v}}{10^8}$ where l is the length of the conductor in cms., B the flux-density in lines per sq. cm., and $\hat{v}$ the velocity of the conductors. But we also know that $\hat{v} = pr$ or $2\pi nr$ if r is the radius of the crank or distance of the conductors from the axis. Hence, if there are Z conductors in the whole coil, the maximum E.M.F. $\hat{\mathrm{E}} = \frac{lBZ \times 2\pi nr}{10^8} = 2rlB \times \frac{\pi nZ}{10^8}$.

But $2rlB = N$, the total number of lines which pass through the coil when in the vertical plane, and hence—

$$\hat{\mathrm{E}} = \pi \frac{nZN}{10^8} \quad \text{and} \quad \bar{\mathrm{E}} = \frac{\hat{\mathrm{E}}}{\sqrt{2}} = \frac{\pi}{\sqrt{2}} \frac{nZN}{10^8} = 2{\cdot}22 \frac{nZN}{10^8}$$

which is the simplest formula for the E.M.F. of an alternator.

If we compare this formula with that for a two-pole direct-current machine, namely, $E = \frac{nZN}{10^8}$, it would appear as if the alternator were more than twice as effective as the D.C. machine. This is, however, easily explained. In the case just taken all the conductors are in series, whereas in the two-pole D.C. armature there are two sets of conductors in parallel. This accounts, therefore, for a coefficient of 2, while the remaining factor 1·11 is due to the effective E.M.F. of the alternator being the root mean square of the E.M.F. in the coil; while in the direct-current armature the same conductors are distributed all round a cylindrical core, and the E.M.F. depends on the ordinary average or mean value of the voltages induced in the conductors in various positions. It is, in fact, the same as the 'form factor' dealt with on p. 100.

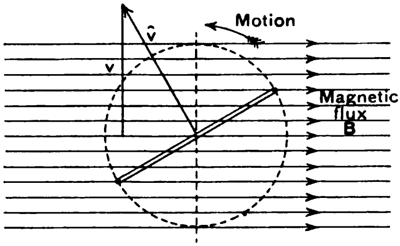

FIG. 65.—Coil rotating in magnetic field.

We may therefore take it that the E.M.F. wave of a simple rotating coil in a uniform field is an exact sine curve, and modern alternators are always designed so as to produce the same wave form as far as possible. We have now to find what voltage is required to produce a simple harmonically varying current in a circuit having capacity, resistance, or inductance, and this will be found perfectly easy.

Quantity, Current, and Electrical Acceleration.—Suppose we have a simple harmonic current represented by the full line curve *C* on the right in Fig. 66. Then we know that this can be represented by a crank of length $\hat{C}$ rotating uniformly with a number of revolutions per second equal to the frequency of the current, and in the horizontal position at the moment the current is zero. Now we saw in Chapters II and III that the quantity and electrical acceleration bear exactly the same relation to the current as mechanical displacement and acceleration bear to the mechanical velocity. But when we are given a piston moving with a simple harmonically varying velocity of which $\hat{v}$ is its maximum value and *n* its frequency, we know that

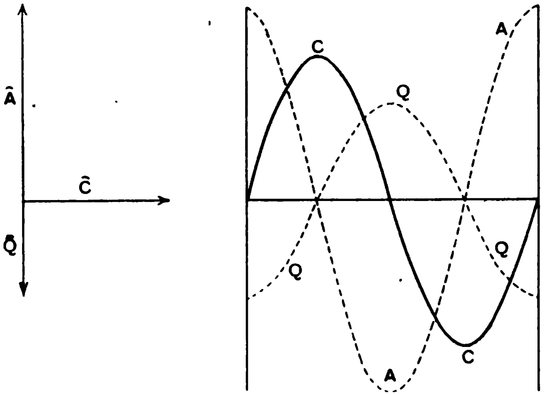

FIG. 66.—Current, quantity, and electrical acceleration.

$\hat{s} = \frac{\hat{v}}{p}$ and lags 90°, or $= \frac{\hat{v}}{p} \urcorner 90°$ (Equation 12, p. 83)

and $\hat{a} = p\hat{v}$ and leads 90°, or $= p\hat{v} \angle 90°$ (Equation 13, p. 83)

Hence, without any further proof, we may write for the current—

$\hat{Q} = \frac{\hat{C}}{p}$ and lags 90° behind $\hat{C}$, or $= \frac{\hat{C}}{p} \urcorner 90°$

$\hat{A} = p\hat{C}$ and leads 90° in front of $\hat{C}$, or $= p\hat{C} \angle 90°$

In fact the quantity, current, and electrical acceleration can be represented by three rotating cranks, and their variation by the three simple harmonic curves in Fig. 66.

Voltage required to produce Current.—We are now able to write down at once the voltage required to pass an alternate current of any given strength through either a condenser, a resistance, or an inductance. For in Table II we had the three relations $V_1 = \frac{1}{K}Q$, $V_2 = rC$, and $V_3 = LA$ for the voltage required in each case. Hence, for a condenser in which the quantity varies in a simple harmonic manner of maximum value $\hat{Q}$, the E.M.F. required to produce this result must be proportional to Q at every instant and have a maximum value $\hat{V}_1 = \frac{1}{K}\hat{Q}$. It must therefore rise and fall with Q or be in phase with it, and since $\hat{Q} = \frac{\hat{C}}{p}$ and lags 90° behind the current, $\hat{V}_1 = \frac{1}{Kp}\hat{C}$ and lags 90° behind the current also. For a resistance, $V_2 = rC$, hence $\hat{V}_2 = r\hat{C}$ and is in phase with $\hat{C}$; while for an inductance, $V_3 = LA$, hence $\hat{V}_3 = L\hat{A} = Lp\hat{C}$ and leads 90° in front of $\hat{C}$. Hence, to produce a current having a maximum value $\hat{C}$ of frequency n periods per second, we have—

$$\text{For condenser } \hat{V}_1 = \frac{1}{Kp}\hat{C} \overline{\diagdown} 90^\circ \quad . \quad . \quad . \quad . \qquad (18)$$

$$\text{,, resistance } \hat{V}_2 = r\hat{C} \text{ in phase with } \hat{C} \quad . \quad . \qquad (19)$$

$$\text{,, inductance } \hat{V}_3 = Lp\hat{C} \angle 90^\circ \quad . \quad . \quad . \quad . \qquad (20)$$

But these voltages and currents are all simple harmonic, and consequently their effective values are obtained by dividing their maximum values by $\sqrt{2}$. Hence, if we divide each of these equations through by $\sqrt{2}$, we have—

$$\text{For condenser } \bar{V}_1 = \frac{1}{Kp}\bar{C} \overline{\diagdown} 90^\circ \quad . \quad . \quad . \quad . \qquad (21)$$

$$\text{,, resistance } \bar{V}_2 = r\bar{C} \text{ in phase} \quad . \quad . \quad . \quad . \qquad (22)$$

$$\text{,, inductance } \bar{V}_3 = Lp\bar{C} \angle 90^\circ \quad . \quad . \quad . \quad . \qquad (23)$$

These three equations therefore give us the voltage, as measured by an ordinary alternate-current voltmeter, required to send a given effective current in amperes through either a condenser, a resistance, or an inductance. It should also be perfectly clear why the voltage and current differ in phase in the case of the capacity and inductance. There is never any difference of phase between the voltage and the effect it produces, but the effect is a displacement in the case of a condenser, and an acceleration in the case of an inductance. The voltage is in phase in each case with the effect produced, and the lag or lead is simply due to the properties of simple harmonic motion, and the lag or lead of the displacement or acceleration with respect to the current.

Current produced by a given Voltage.—In order to obtain the current when a given maximum or effective voltage is applied to a condenser, resistance, or inductance, we have only to invert the equations already obtained. Taking the condenser, we have $\hat{V}_1 = \frac{1}{Kp}\hat{C}$ and lags 90°. Hence, it is clear that $\hat{C} = Kp\hat{V}_1$ and is earlier than $\hat{V}_1$ by 90° or leads by 90° in front of $\hat{V}_1$. If we now suppose that a voltage of maximum value $\hat{V}$ or effective value $\bar{V}$ is connected to a condenser, resistance, or inductance, and call the respective currents produced $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$, or $\bar{C}_1$, $\bar{C}_2$, and $\bar{C}_3$ for their maximum and their effective values respectively, we must have by similar inversion of all the equations—

$$\text{For condenser } \hat{C}_1 = Kp\hat{V} = \frac{\hat{V}}{\frac{1}{Kp}}, \text{ or } \bar{C}_1 = Kp\bar{V} = \frac{\bar{V}}{\frac{1}{Kp}} \angle 90^\circ \quad (24)$$

$$\text{,, resistance } \hat{C}_2 = \frac{\hat{V}}{r}, \qquad \text{or } \bar{C}_2 = \frac{\bar{V}}{r} \text{ in phase} \quad . \quad . \quad (25)$$

$$\text{,, inductance } \hat{C}_3 = \frac{\hat{V}}{Lp}, \qquad \text{or } \bar{C}_3 = \frac{\bar{V}}{Lp} \Gamma 90^\circ \quad . \quad . \quad (26)$$

It will be noticed that, in each of the last three equations, the effective current is obtained by dividing the effective voltage by a quantity which is constant if the frequency is constant. Hence we may say in a sense that, instead of the ordinary Ohm's law $C = \frac{V}{r}$ for direct current, there are three Ohm's laws for alternate currents—$\bar{C}_1 = \frac{\bar{V}}{\frac{1}{Kp}}$ for a condenser, $\bar{C}_2 = \frac{\bar{V}}{r}$ for a resistance (like the ordinary Ohm's law), and $\bar{C}_3 = \frac{\bar{V}}{Lp}$ for an inductance. The quantities $\frac{1}{Kp}$ and Lp consequently play the same part to alternating currents (so far as magnitude is concerned) as resistance does, and may evidently therefore be reckoned in ohms. Owing, however, to the current produced being displaced in phase, and to their being dependent on the frequency, these quantities are not spoken of as resistances but as *reactances*. The quantity $\frac{1}{Kp}$ for the condenser, where K is the capacity in farads and $p = 2\pi n$, is called the *capacity reactance* of the condenser in ohms, at the frequency n; and similarly Lp, where L is the inductance in henries, is the *inductive reactance* or simply the *reactance* of the inductance at the frequency n. Hence our Ohm's law for alternating currents may be expressed as follows:

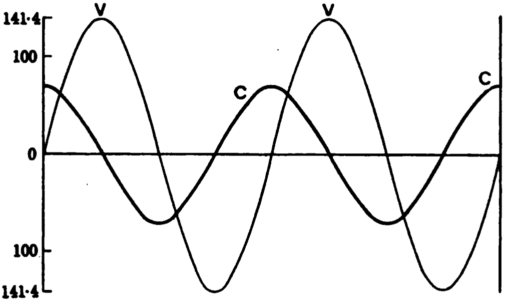

FIG. 67.—P.D. and current curves for condenser.
$\bar{V} = 100$; $\frac{1}{Kp} = 2$ ohms.

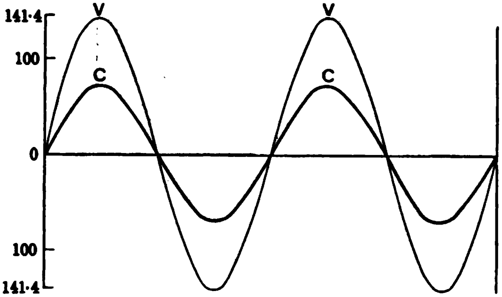

FIG. 68.—P.D. and current curves for resistance.
$\bar{V} = 100$; $r = 2$ ohms.

The effective current produced by a simple harmonic voltage applied to a condenser, resistance, or inductance is obtained by dividing the effective P.D. in volts by the resistance or reactance of the circuit in ohms. The current in the case of the condenser leads 90° in front of the P.D., and in the case of the inductance lags 90° behind it. It is customary to take the inductive reactance as positive and the capacity reactance as negative, for reasons which will be appreciated later.

Figs. 67, 68, and 69 show P.D. and current curves for a condenser, resistance, and inductance respectively in which the effective value of the P.D. is 100 volts, and the reactance or resistance is in each case 2 ohms.

We may now summarise our conclusions in the following table:

TABLE X.—ALTERNATE-CURRENT RELATIONS

VOLTAGE REQUIRED FOR GIVEN SIMPLE HARMONIC CURRENT.					
Circuit.	**Fundamental Equation.**	**Simple Harmonic Properties.**	**Maximum Value.**	**Effective Value.**	**Phase.**
Condenser	$V_1 = \frac{1}{K}Q$	$\hat{Q} = \frac{\hat{C}}{p} \ulcorner 90^\circ$	$\hat{V}_1 = \frac{1}{Kp}\hat{C}$	$\bar{V}_1 = \frac{1}{Kp}\bar{C}$	Lags 90°
Resistance	$V_2 = rC$		$\hat{V}_2 = r\hat{C}$	$\bar{V}_2 = r\bar{C}$	In phase
Inductance	$V_3 = LA$	$\hat{A} = p\hat{C} \angle 90^\circ$	$\hat{V}_3 = Lp\hat{C}$	$\bar{V}_3 = Lp\bar{C}$	Leads 90°
CURRENT OBTAINED WITH GIVEN SIMPLE HARMONIC VOLTAGE.					
Circuit.	**Fundamental Equation.**	**Simple Harmonic Properties.**	**Maximum Value.**	**Effective Value.**	**Phase.**
Condenser	$Q = KV$	$\hat{C} = p\hat{Q} \angle 90^\circ$	$\hat{C}_1 = Kp\hat{V} = \frac{\hat{V}}{\frac{1}{Kp}}$	$\bar{C}_1 = Kp\bar{V} = \frac{\bar{V}}{\frac{1}{Kp}}$	Leads 90°
Resistance	$C = \frac{V}{r}$		$\hat{C}_2 = \frac{\hat{V}}{r}$	$\bar{C}_2 = \frac{\bar{V}}{r}$	In phase
Inductance	$A = \frac{V}{L}$	$\hat{C} = \frac{\hat{A}}{p} \ulcorner 90^\circ$	$\hat{C}_3 = \frac{\hat{V}}{Lp}$	$\bar{C}_3 = \frac{\bar{V}}{Lp}$	Lags 90°

Numerical Example.—As a close to this section we may take the following numerical example:

What current will be produced with 100 volt 50 ~ circuit in the following cases: (a) a condenser of 20 microfarads (20×10^{-6} farads), (b) a resistance of 50 ohms, and (c) an inductance of ·05 henry? Here $\bar{V} = 100$, $p = 2\pi \times 50 = 314$.

(a) Capacity reactance $= \frac{1}{Kp} = \frac{1}{20 \times 10^{-6} \times 314} = 159{\cdot}4$ ohms.

(b) Resistance $r = 50$ ohms.

(c) Reactance $= Lp = {\cdot}05 \times 314 = 15{\cdot}7$ ohms.

Hence current in condenser $= \frac{100}{159{\cdot}4} = {\cdot}628$ effective amperes, leading 90° in front of voltage.

Current in resistance $= \frac{100}{50} = 2$ effective amperes, in phase.

Current in inductance $= \frac{100}{15{\cdot}7} = 6{\cdot}37$ effective amperes, lagging 90° behind voltage.

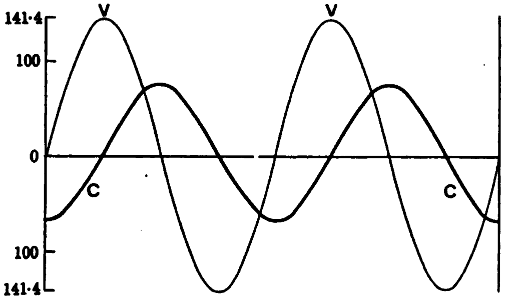

FIG. 69.—P.D. and current curves for inductance.
$\overline{V} = 100$; $Lp = 2$ ohms.

The general correctness of the relations which we have found above can be well realised by a few simple mechanical illustrations, using a spring or elastic cord in place of a condenser, a paddle in water, or a fan, the friction of which imitates the resistance, and a weight in place of an inductance. In the case of the fan or paddle (Fig. 70), if it is swung backwards and forwards, one can easily feel that the force is greatest in the middle of the swing when the velocity is greatest, *i.e.* the force and velocity are in phase, like the volts and amperes in a resistance.

Turning to the spring, the best type to use is a straight strip or a stretched cord, as in Fig. 71. If this is gripped and moved backwards and forwards in a more or less regular manner, the velocity will of course be greatest at the centre of the swing, but the greatest force in the same direction will be required a quarter of a period later, at the end of the swing, when the spring is stretched by its greatest amount. Hence the velocity corresponding to the current leads 90° in front of the force corresponding to the voltage, as we saw for the condenser.

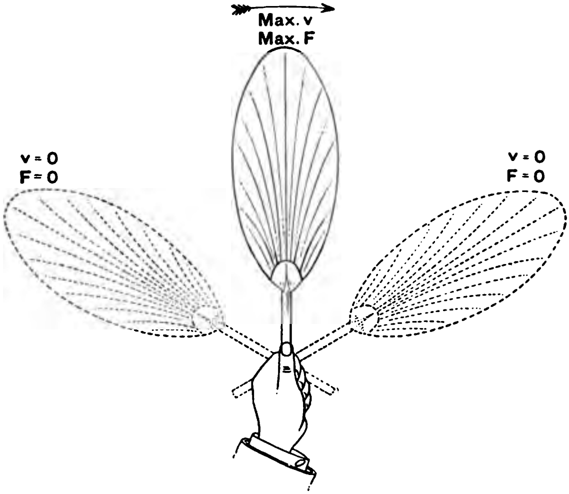

FIG. 70.—Motion of fan.

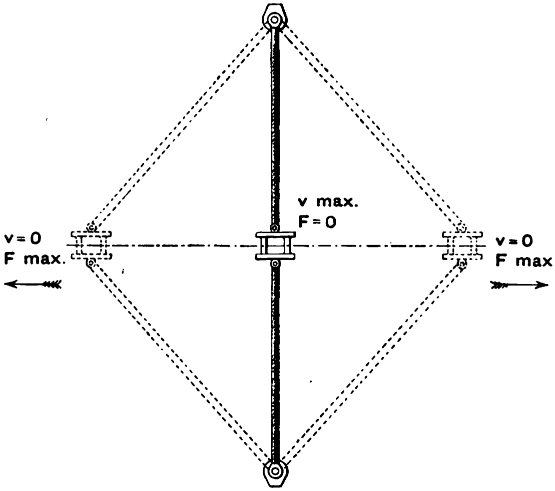

FIG. 71.—Motion of spring.

Finally, if we have heavy weights, such as dumb-bells, and swing them backwards and forwards very rapidly (Fig. 72), we shall distinctly feel that the greatest force in a given direction must be exerted at the beginning of the swing, whereas the greatest velocity in the same direction does not take place till the middle of the swing. The velocity therefore lags a quarter period or 90° behind the force, as the current lags behind the P.D. in the case of an inductive circuit. A heavy iron gate with well-oiled hinges shows this even better, as the only force to be exerted in this case is that required to accelerate the mass, owing to the weight being supported by the hinges.

It is therefore possible to combine the study of alternate currents with physical development or calisthenics, which ought to make it extremely popular. The vertical cords of the well-known Sandow exerciser are admirably suited for the spring, and this in combination with a pair of moderate-weight dumb-bells and a large fan, forms a complete set of apparatus for the study of elementary alternate-current phenomena.

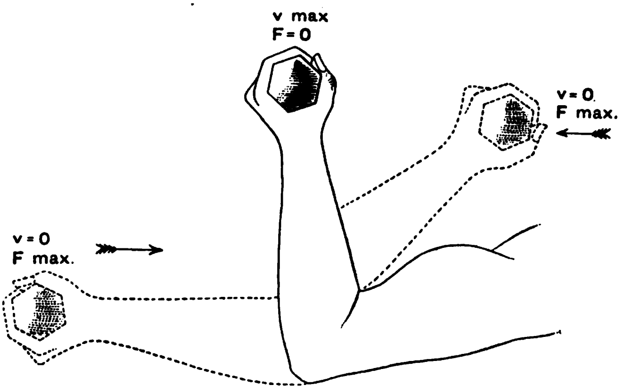

FIG. 72.—Motion of mass.

The truck model shown in Fig. 40 will show the above effect more perfectly. For this purpose the model is employed with two flasks of water attached to the two vertical cords, and the water is driven from one of these flasks to the other by means of a cylinder and piston. The movement of the piston to right or to left causes a weighting down of the corresponding side and a force tending to move the truck and cord in the corresponding direction. When the piston is moved by a uniformly rotating crank, the driving force is varied in an exact simple-harmonic manner, and it can be seen that when the spring or 'condenser' is employed, the displacement of the truck is exactly in phase with that of the piston or driving force, and its greatest velocity obviously occurs a quarter of a period earlier. On releasing the spring and attaching the sail, the truck oscillates forwards and backwards, but its maximum displacement occurs a quarter of a period later than the maximum force, which shows that the maximum velocity is in phase with the force.

Lastly, if the truck is weighted and the friction removed, it will be found that the displacement of the truck is exactly opposite to that of the piston, or lags 180° behind the force, and since the velocity leads 90° in front of the displacement, it lags 90° behind the force, as in the case of the inductance. By using a speed indicator on the cord, which forms a mechanical ammeter, the phase-relations of force and speed can be directly seen.

CHAPTER X

COMBINATIONS OF RESISTANCE, INDUCTANCE, AND CAPACITY IN SERIES

In the last chapter we have found the maximum and effective voltages required to produce an alternate current of given maximum or effective value in a circuit having any one of the three quantities, resistance, inductance, or capacity only. We now proceed to consider what the relations must be when we have combinations of these quantities in series with one another.

In Fig. 73 we have a condenser of capacity K in series with a non-inductive resistance r, and a coil having an inductance of L henries and assumed to be free from resistance. Then if a current C traverses the three, we know by the rules given in the last chapter how to calculate the three voltages V_1, V_2, and V_3 at the terminals of the condenser, resistance, and inductance separately. Fig. 74 shows the curves of the three voltages when a simple harmonic current traverses the circuit. Now if V is the total P.D. across the combination, we know that at every instant $V = V_1 + V_2 + V_3$, or the total voltage is the sum of the voltages required to charge the condenser, force the current against the resistance, and accelerate the current through the inductance. If, therefore, we add the ordinates of the three curves together, we get the curve of total voltage V across the combination.

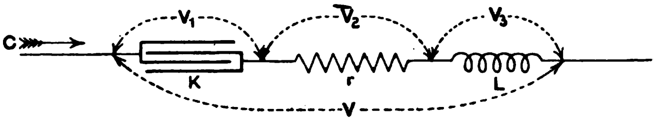

FIG. 73.—Condenser, resistance, and inductance in series.

If the current is simple harmonic, each of the curves of

V_1, V_2, and V_3 will be simple harmonic. But in Chapter VIII, on the combination of simple harmonic motions, we saw that the sum of any number of simple harmonic motions of the same frequency was itself a simple harmonic motion, the crank or vector of which was the sum of the cranks or vectors of the component motions, added like forces.

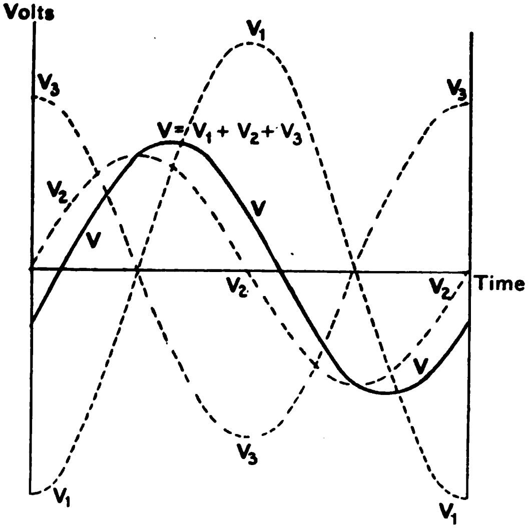

FIG. 74.—Voltage curves for condenser, resistance, and inductance.

If $\hat{C}$ is the maximum value of the current and $p = 2\pi n$, then we have (Equations 18, 19, and 20, p. 113)—

For condenser, $\hat{V}_1 = \frac{1}{Kp}\hat{C}$ and lags 90° behind current

For resistance, $\hat{V}_2 = r\hat{C}$ in phase with current

For inductance, $\hat{V}_3 = Lp\hat{C}$ and leads 90° in front of current.

Hence, if in Fig. 75 we represent the current by a horizontal crank or vector, we must represent V_1 as a vector of length $\frac{1}{Kp}\hat{C}$ vertically downwards, or a right angle behind $\hat{C}$; V_2 as a vector of length $r\hat{C}$ horizontally along the direction of the current vector; and V_3 as a vector of length $Lp\hat{C}$ at right angles to $\hat{C}$ vertically upwards. On adding these three together like forces, a crank or vector representing the total voltage V will be obtained. Hence, if we go down $\hat{V}_1$ and add $\hat{V}_2$ horizontally to it, and finally put $\hat{V}_3$ vertically upwards from the end of $\hat{V}_2$, then the vector $\hat{V}$ from the commencement of $\hat{V}_1$ to the end of $\hat{V}_3$ must be the crank or vector of the total voltage, and its length to scale must be the maximum value of this total voltage; while its angle of inclination ϕ to the horizontal or $\hat{C}$ shows the difference of phase between it and the current. It is easily seen, therefore, that the total voltage is given by the hypothenuse or longest side of a right-angled triangle, of which the sides are $\hat{V}_2 = r\hat{C}$ and $\hat{V}_3 - \hat{V}_1 = \left(Lp - \frac{1}{Kp}\right)\hat{C}$, and since the value of this hypothenuse is the square root of the sum of the squares of the other two sides,

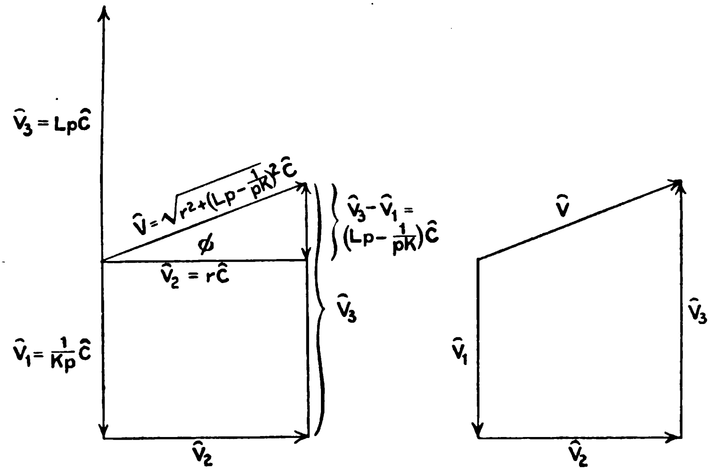
$\hat{V}_3 = Lp\hat{C}$
$\hat{V} = \sqrt{r^2+(Lp-\frac{1}{pK})^2}\hat{C}$
ϕ
$\hat{V}_2 = r\hat{C}$
$\hat{V}_3 - \hat{V}_1 = (Lp-\frac{1}{pK})\hat{C}$
$\hat{V}_3$
$\hat{V}_1 = \frac{1}{Kp}\hat{C}$
$\hat{V}_2$
$\hat{V}$
$\hat{V}_1$
$\hat{V}_3$
$\hat{V}_2$

FIG. 75.—Vector diagram of maximum voltages.

$$\hat{V} = \sqrt{\hat{V}_2^2 + (\hat{V}_3 - \hat{V}_1)^2} = \sqrt{r^2 + \left(Lp - \frac{1}{Kp}\right)^2}\hat{C}.$$

This method and figure is, in fact, exactly like that which we employed in the engine problem on p. 106, except that the second vector $\hat{V}_2$ has been taken as horizontal in this case, instead of the first vector $\hat{F}_1$ in the engine problem, and the whole figure is therefore turned through a right angle. This turning does not, however, affect the formula for the magnitude of the total P.D.

The angle of lead ϕ of the P.D. in front of the current can be measured by a protractor, or can be calculated from the formula $\tan \phi = \frac{\hat{V}_3 - \hat{V}_1}{\hat{V}_2} = \frac{Lp - \frac{1}{Kp}}{r}$. This does not correspond exactly to the formula for the angle in the engine problem, as we are here reckoning the phase of the P.D. with respect to the current which corresponds to the velocity of the piston, whereas before it was reckoned with respect to the displacement.

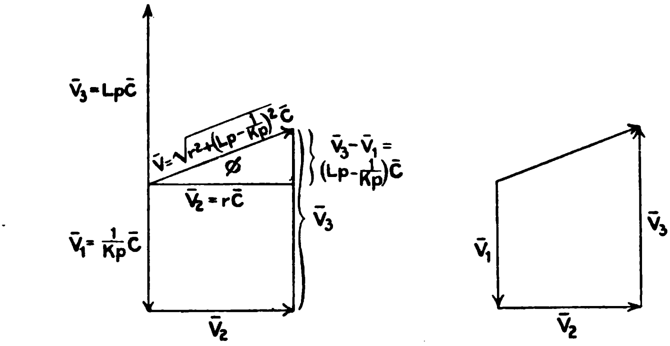
$\bar{V}_3 = Lp\bar{C}$
$\bar{V} = \sqrt{r^2 + (Lp - \frac{1}{Kp})^2}\,\bar{C}$
ϕ
$\bar{V}_2 = r\bar{C}$
$\bar{V}_1 = \frac{1}{Kp}\bar{C}$
$\bar{V}_3 - \bar{V}_1 = (Lp - \frac{1}{Kp})\bar{C}$
$\bar{V}_3$
$\bar{V}_2$
$\bar{V}_1$
$\bar{V}_3$
$\bar{V}_2$

FIG. 76.—Vector diagram of effective voltages.

The P.D. required to pass a current through a combination of any amount of resistance, inductance, and capacity in series can thus be found. But the result we have obtained is the *maximum* value of the total P.D., instead of the *effective* value, which is what we really require, and this has been obtained by taking the *maximum* values of the current and of all the separate voltages. Now suppose we draw the whole diagram again as in Fig. 76, starting with the effective current $\bar{C}$, and marking off the three effective voltages $\bar{V}_1 = \frac{1}{Kp}\bar{C} \ulcorner 90^\circ$, $\bar{V}_2 = r\bar{C}$ in phase, and $\bar{V}_3 = Lp\bar{C} \angle 90^\circ$, as we saw in the last chapter. Since the current is supposed to be simple harmonic, its effective value $\bar{C}$ is $\frac{1}{\sqrt{2}}$ times $\hat{C}$, and similarly $\bar{V}_1$, $\bar{V}_2$, and $\bar{V}_3$ are each of them obtained by dividing V_1, V_2, and $\hat{V}_3$ by $\sqrt{2}$. The diagram (Fig. 76) is therefore precisely the same as the previous one (Fig. 75), except that each line in the second diagram is ·707 or

$\frac{1}{\sqrt{2}}$ times the length of the corresponding line in the first. Consequently, since the whole of the second diagram has been reduced from the first in the ratio $\frac{1}{\sqrt{2}}$, the resultant vector from the beginning of $\bar{V}_1$ to the end of $\bar{V}_3$ is evidently reduced in the same ratio and must equal $\frac{V}{\sqrt{2}}$, which is the effective value of V, or $\bar{V}$. Hence, if we start with the effective value of the current and mark off all our separate voltages as effective values, we obtain the effective value of the total P.D., which gives us what we require without any introduction of $\sqrt{2}$. This result would evidently apply generally, whether the vectors were at right angles or not, and we have the following very important rule:

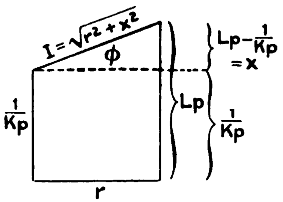

FIG. 77.—Vector diagram of resistance and reactance.

The **effective** *value of the sum or resultant of any number of simple harmonically varying quantities is obtained by adding together the* **effective** *values of the separate quantities like forces or vectors.*

Although this gives us all we require for actual solution of the problem, there is yet a further simplification which can be introduced when we want to obtain convenient formulæ. The three effective voltages $\bar{V}_1 = \frac{1}{Kp}\bar{C} \overline{\diagdown} 90^\circ$, $\bar{V}_2 = r\bar{C}$, and $\bar{V}_3 = Lp\bar{C} \angle 90^\circ$ are each of them proportional to $\bar{C}$, and hence the final effective P.D. $\bar{V}$ must be proportional to $\bar{C}$. If, therefore, we find the effective P.D. required to pass one effective ampere through the combination, the result when multiplied by the effective value of any other current will give us the total effective P.D. required in that case.

In the diagram (Fig. 77) we have therefore drawn the effective voltages $\bar{V}_1$, $\bar{V}_2$, and $\bar{V}_3$ for a current of one effective ampere. In this case, therefore, $\bar{V}_1 = \frac{1}{Kp}\bar{C} \overline{\diagdown} 90^\circ = \frac{1}{Kp} \overline{\diagdown} 90^\circ$ simply, since $\bar{C} = 1$; *i.e.* $\bar{V}_1$ is simply the capacity reactance of the condenser in ohms set off as a vector vertically downwards. Similarly $\bar{V}_2 = r\bar{C} = r$ only, or is the resistance in ohms, set off horizontally. Finally $\bar{V}_3 = Lp\bar{C} \angle 90° = Lp \angle 90°$, or is the inductive reactance of the coil in ohms, set off vertically upwards. When these are added together as vectors we get the effective P.D. in volts required to pass one effective ampere through the whole combination, and this is represented by the hypothenuse I of a right-angled triangle, of which the horizontal side is the resistance r, and the vertical side $Lp - \frac{1}{Kp}$; the latter is called the total reactance and is represented by the symbol x. We thus have—

$$\mathrm{I} = \sqrt{r^2 + x^2} = \sqrt{\text{resistance}^2 + \text{reactance}^2}$$
$$= \sqrt{r^2 + \left(Lp - \frac{1}{Kp}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

and $$\tan\phi = \frac{x}{r} = \frac{\text{reactance}}{\text{resistance}} = \frac{Lp - \frac{1}{Kp}}{r} \text{ as before.} \qquad (28)$$

Now what is the quantity I? It is the effective P.D. in volts for one effective ampere, and hence the effective P.D. for two effective amperes will be 2I volts, and for $\bar{C}$ effective amperes $\bar{V} = \bar{C}\mathrm{I}$ volts, leading in front of C by the angle ϕ, for which $\tan\phi = \frac{x}{r}$. Conversely we must have $\bar{C} = \frac{\bar{V}}{\mathrm{I}} = \frac{\bar{V}}{\sqrt{r^2 + x^2}}$ and lags by the angle ϕ behind $\bar{V}$. Now, if we have a circuit having resistance only, $\mathrm{I} = \sqrt{r^2 + x^2} = r$ and $\bar{C} = \frac{\bar{V}}{r}$, while $\tan\phi = \frac{x}{r} = 0$, from which $\phi = 0$ (see Fig. 78), or the current is in phase, as we already know. On the other hand, if we have a circuit having inductance and capacity, without resistance, $\mathrm{I} = \sqrt{r^2 + x^2} = x$, and $\bar{C} = \frac{\mathrm{V}}{x}$, while $\tan\phi = \frac{x}{r} = \frac{x}{0}$, which is infinite or $\phi = 90°$, as is evident from Fig. 79. Consequently—

For resistance only	$\bar{C} = \frac{\bar{V}}{r}$ and is in phase
„ reactance only	$\bar{C} = \frac{\bar{V}}{x} \ulcorner 90°$
„ resistance and reactance	$\bar{C} = \frac{\bar{V}}{\mathrm{I}} \ulcorner \phi$

It is clear that I is a quantity which, like resistance or reactance, can be expressed in ohms. It cannot be called resistance, however, for the current produced in it is not in phase with the P.D., nor can it be called reactance, because the current does not lag 90°, but it is a combination of the two, and represents the total retarding or impeding effect of the combination. It has therefore been termed the *impedance* of the circuit, and we thus have—impedance $= \sqrt{\text{resistance}^2 + \text{reactance}^2}$, *i.e.* it is the hypothenuse of a right-angled triangle of which the resistance and the reactance are the two other sides, and is equal to the resistance or to the reactance respectively, if the circuit contains one or the other of them only. It will be clear from the foregoing why capacity reactance is reckoned negative as was mentioned on p. 116. The total or effective reactance of a circuit $x = Lp - \frac{1}{Kp}$ or the inductive reactance *minus* the capacity reactance. If we speak of capacity reactances as being negative we can say that the total reactance is the algebraic sum of all the reactances in series in a circuit.

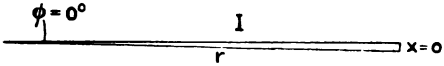

FIG. 78.—Impedance diagram for circuit with very small reactance.

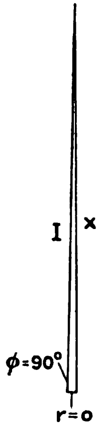

FIG. 79.—Impedance diagram for circuit with very small resistance.

Before taking some numerical examples there is a case of such frequent occurrence that it will be dealt with separately, viz. that of an ordinary inductive or choking coil having resistance and inductance. It is clear that in causing a current to pass through such a coil, a P.D. must be provided enough to overcome its resistance and its reactance, and that therefore this case is equivalent to a resistance in series with an inductance. The reactance $x = Lp - \frac{1}{Kp}$ is therefore equal to Lp only,* and we have $\mathrm{I} = \sqrt{r^2 + x^2} = \sqrt{r^2 + L^2p^2}$ and $\tan\phi = \frac{x}{r} = \frac{Lp}{r}$. Hence, if a P.D. $\bar{V}$ is applied to such a coil, we obtain a current $\bar{C} = \frac{\bar{V}}{\sqrt{r^2 + L^2p^2}}$ lagging by the angle ϕ, for which $\tan\phi = \frac{Lp}{r}$.

Numerical Examples.—We can now work out some practical examples, and we will first take the case of a choking coil having a resistance of 10 ohms and an inductance of ·05 henry, connected to a 100-volt circuit at 50 ∼. Here we have $p = 2\pi n = 314$, and $x = Lp = \cdot05 \times 314 = 15{\cdot}7$ ohms. Hence the impedance is the hypothenuse of a right-angled triangle having sides of $r = 10$ and $x = 15{\cdot}7$ ohms, which can be scaled off as 18·6 ohms. Or $\mathrm{I} = \sqrt{r^2 + x^2} = \sqrt{10^2 + 15{\cdot}7^2} = 18{\cdot}61\omega$, and $\tan\phi = \frac{15{\cdot}7}{10} = 1{\cdot}57$, from which $\phi = 57{\cdot}5^\circ$. Consequently the current $\bar{C} = \frac{\bar{V}}{\mathrm{I}} = \frac{100}{18{\cdot}61} = 5{\cdot}375$ amps. lagging by the angle $\phi = 57{\cdot}5^\circ$.

If the same coil is taken with the same voltage but at different frequencies, we must first take a direct current, for which $C = \frac{V}{r} = 10$ amperes. This is the same thing as taking the alternate-current formula $\bar{C} = \frac{\bar{V}}{\sqrt{r^2 + L^2p^2}}$ and making the frequency or $p = 0$, from which $\mathrm{I} = r$, and $\tan\phi = \frac{Lp}{r} = 0$; hence $C = \frac{V}{r} = \frac{100}{10} = 10$ amps.

If we take a frequency of only 10 ∼, $p = 62{\cdot}8$, and $x =$

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* There might possibly be some confusion here, as if $x = Lp - \frac{1}{Kp}$ and $K = 0$, $\frac{1}{Kp} = \infty$ instead of zero. But a condenser of no capacity is evidently equivalent to a dead break instead of a perfect conductor. A condenser of infinitely large capacity is equivalent to a perfect conductor, like a spring of infinitely large yield for a given force. Such a spring would have no resisting effect.

TABLE XI.—COMBINATIONS OF CAPACITY, RESISTANCE, AND INDUCTANCE IN SERIES

Nature of Circuit.	P.D. for given Current.	Current for given P.D.
Capacity	$\bar{V} = \frac{1}{Kp}\bar{C} \ulcorner 90^\circ$	$\bar{C} = Kp\bar{V} \angle 90^\circ$
Resistance	$\bar{V} = r\bar{C}$ in phase	$\bar{C} = \frac{\bar{V}}{r}$ in phase
Inductance	$\bar{V} = Lp\bar{C} \angle 90^\circ$	$\bar{C} = \frac{\bar{V}}{Lp} \ulcorner 90^\circ$
Capacity and Inductance	$\bar{V} = x\bar{C}$ leading or lagging 90°	$\bar{C} = \frac{\bar{V}}{x}$ lagging or leading 90°
	where $x = Lp - \frac{1}{Kp}$	
Capacity and Resistance	$\bar{V} = \mathrm{I}\bar{C} \ulcorner \phi$	$\bar{C} = \frac{\bar{V}}{\mathrm{I}} \angle \phi$
	where $\mathrm{I} = \sqrt{r^2 + \frac{1}{K^2p^2}}$, and $\tan\phi = \frac{1}{Krp}$	
Resistance and Inductance	$\bar{V} = \mathrm{I}\bar{C} \angle \phi$	$\bar{C} = \frac{\bar{V}}{\mathrm{I}} \ulcorner \phi$
	where $\mathrm{I} = \sqrt{r^2 + L^2p^2}$, and $\tan\phi = \frac{Lp}{r}$	
Capacity, Resistance, and Inductance	$\bar{V} = \mathrm{I}\bar{C} \ulcorner \phi$	$\bar{C} = \frac{\bar{V}}{\mathrm{I}} \angle \phi$
	where $\mathrm{I} = \sqrt{r^2 + x^2}$, $x = Lp - \frac{1}{Kp}$, and $\tan\phi = \frac{x}{r}$	

$\cdot05 \times 62\cdot8 = 3\cdot14$ ohms; hence $\mathrm{I} = \sqrt{10^2 + 3\cdot14^2} = 10\cdot48$, $\tan\phi = \frac{3\cdot14}{10} = \cdot314$, $\phi = 17\cdot44^\circ$, and $C = \frac{100}{10\cdot48} = 9\cdot55$ amps.

Proceeding in this way, we can draw up a table as follows:

TABLE XII.—INDUCTIVE COIL ON ALTERNATE-CURRENT CIRCUIT

V = 100, r = 10 ohms, L = ·05 henry.

Frequency ∼ per Sec.	$p = 2\pi n$.	$x = Lp$ Ohms.	$I = \sqrt{r^2 + x^2}$ Ohms.	$C = \frac{V}{I}$ Amperes.	Tan $\phi = \frac{x}{r}$.	ϕ Degrees.
0	0	0	10	10	0	0
10	62·8	3·14	10·48	9·55	·314	17° 26′
20	125·6	6·28	11·81	8·47	·628	32° 8′
30	188·4	9·42	13·75	7·28	·942	43° 18′
40	251·2	12·56	16·05	6·23	1·256	51° 28′
50	314	15·7	18·61	5·37	1·57	57° 30′
60	376·8	18·84	21·33	4·69	1·884	62° 2′
80	502·4	25·12	27·05	3·70	2·512	68° 18′
100	628	31·4	33·0	3·03	3·14	72° 21′
150	942	47·1	48·2	2·075	4·71	78° 1′
200	1256	62·8	63·6	1·57	6·28	80° 6′

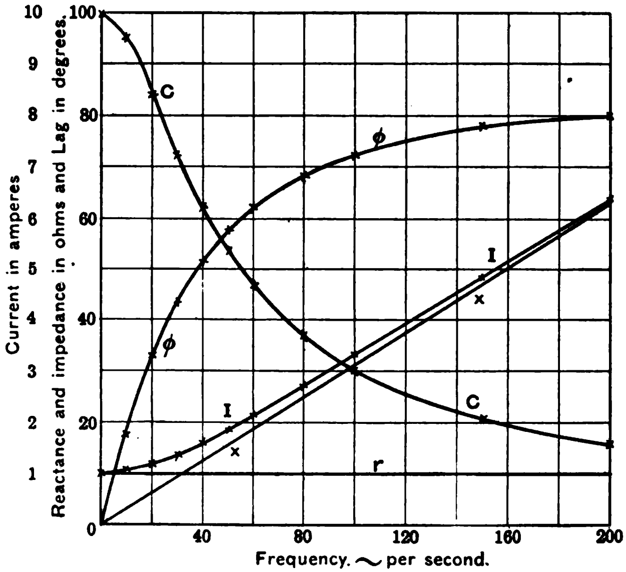

FIG. 80.—Variation of current, &c., with frequency, in choking coil.

The variation of the current reactance, impedance and angle of lag with frequency in this case is well seen in the curves (Fig. 80). The impedance, from being equal to the resistance at zero frequency, increases as the frequency rises, and becomes nearly equal to the reactance at the higher frequencies, while the current gets less and the lag greater.

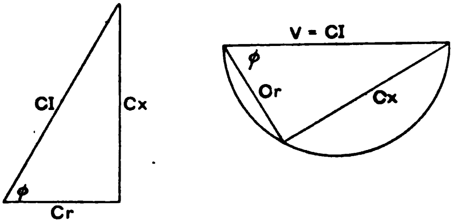

FIG. 81.—Semicircle diagram.

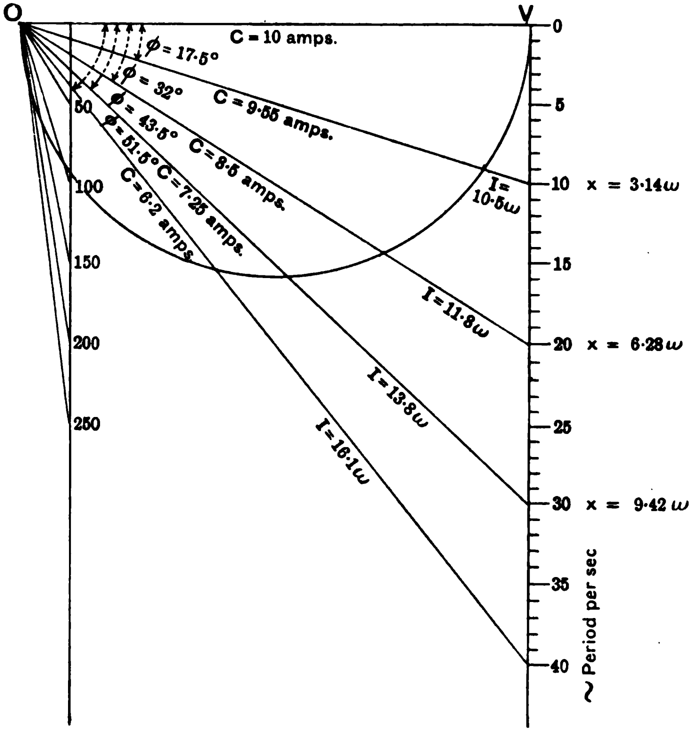

FIG. 82.—Graphical method of finding current in choking coil.

In Fig. 82 is given a neat graphical method of rapidly solving the same problem. Whatever be the values of the resistance and reactance in a circuit, the resistance drop *Cr* and

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the reactance drop Cx together produce the total drop CI, which is the total voltage at the terminals. If, therefore, this voltage is constant for all the frequencies, the resultant CI is constant and equal to V. Now Cr and Cx must always be at right angles, since one is in phase and the other in quadrature with the current. But it is shown in Euclid (III, 31), and can be easily tested by measurement, that if we have a semicircle such as Fig. 81, then if any point is taken on its circumference and straight lines are drawn thence to the two ends of the diameter, these two lines will be at right angles whatever be the point chosen; or conversely, if two lines drawn from the ends of a fixed line are at right angles to one another, they must lie in a semicircle described on the fixed line as diameter. Consequently, if we lay down a fixed line of a length which represents V to a convenient scale, find its middle point, and draw a semicircle on the line, we know that, whatever be the frequency, Cr and Cx must meet on this semicircle. But as the resistance r is constant, Cr is proportional to C, and hence as the frequency varies the current vector must move along a semicircle. In order to find where it is, however, for any particular frequency we have only to remember that $\tan \phi = \frac{x}{r}$, or that ϕ is the angle of a right-angled triangle, of which r is the adjacent and x the opposite side. As r is fixed it can also be represented by a fixed horizontal line, and the values of x for various frequencies can then be plotted along a perpendicular line as in Fig. 82. By joining these points to O in each case the new side is the impedance, and the angle between it and the horizontal is ϕ. Hence this line is at the angle which Cr makes with V, and the portion inside the semicircle is the resistance drop Cr, and is thus proportional to the current.

In our case this enables the result to be very quickly obtained. We first lay down a line 10 cm. or 100 mm. long, which, at 1 cm. to the ohm, represents the 10 ohms of resistance of the coil, and also at 1 mm. to the volt the 100 volts P.D. on the circuit. (The scale in Fig. 82 is about two-thirds of this.) A perpendicular is now drawn on the end of this line, and since the reactance at a frequency of 10 ~ is $3{\cdot}14\omega$ we mark off 3·14 cm. as the reactance for this frequency, and mark it with 10, representing the frequency. As the reactance is proportional to the frequency we may proceed to mark off twice, three times, &c., this value along the vertical line, as the reactances for frequencies of 20, 30, &c., ~ per sec. respectively.

If these points are then joined to O, the length of each of these lines in cm. is $\sqrt{r^2 + x^2} =$ the impedance in ohms, and the angle ϕ is the angle of lag. But if we now draw a semicircle on OV, the length in mm. of the part of the line inside the semicircle is the resistance drop in volts, and since $r = 10$ ohms the current is one-tenth of the resistance drop. Hence one-tenth of the length in mm., which equals its length in cm., is the current itself in amperes. Consequently we can scale off instantly from the diagram the impedance, lag, and current for each frequency, and it will be seen that the values correspond closely with those obtained by calculation. With the scale we have chosen the reactance soon gets too long, but that is simply got over by drawing another vertical line at 1 cm. instead of 10 cm. from O, and marking off the reactances to one-tenth the scale, so that 50 ∼ on this new vertical line is represented by the same distance as 5 ∼ on the old one, as is evident from the proportionality of the sides in similar triangles.

Tracing of A.C. Waves in Inductive Circuits by the Planimeter. Wherever a circuit consists of resistance and inductance only, the current produced by a P.D. varying in any manner whatever can be obtained by Dr. Sumpner's method described on p. 48, or by the planimeter based on it. As we there saw, the actual variation of the current is got by drawing a curve which represents what the current would be if the circuit had resistance only, or $\frac{V}{r}$, which is of course similar to the voltage curve itself, and then drawing lines to points which are ahead of the points on this curve by a time equal to the time-constant of the circuit.

Fig. 33 (B and C) shows the method of obtaining an alternate-current wave in this manner, and the variation of current on switching on an inductive circuit at various instants during the period of an alternating E.M.F. It will be seen that the current at first depends to a considerable extent upon the moment in the period at which it is switched on, but that all the curves unite after a time to form the curve given by our theory, which, of course, only applies to a cyclic or constantly repeating condition.

In the illustration the frequency of the current is supposed to be 50 ∼ per sec. and the time-constant $T = {\cdot}01$ sec. or half the periodic time of the P.D. In Fig. 33C the P.D. curve is of non-sinusoidal form, and it is clearly seen how the inductance smooths out the irregularities of the P.D. wave.

One other example will now be taken, and will be found of even greater interest. We will consider a circuit consisting of a choking coil of 30ω resistance and ·7 henry inductance, coupled in series with a 10-mfd. condenser to a 200-volt circuit at different frequencies. In this case we have—

Resistance $r = 30$ ohms

Inductive reactance $Lp = {\cdot}7p$ ohms

Capacity reactance $\dfrac{1}{Kp} = \dfrac{10^6}{10p} = \dfrac{10^5}{p}$ ohms

Resultant or total reactance $x = Lp - \dfrac{1}{Kp} = {\cdot}7p - \dfrac{10^5}{p}$ ohms

Total impedance $\mathrm{I} = \sqrt{r^2 + x^2}$ ohms

Current $\bar{\mathrm{C}} = \dfrac{\bar{\mathrm{V}}}{\mathrm{I}} = \dfrac{200}{\sqrt{r^2 + x^2}}$ effective amperes

Angle of lag $\tan\phi = \dfrac{x}{r} = \dfrac{x}{30}$

Impedance of coil only $\mathrm{I}_1 = \sqrt{r^2 + L^2p^2}$

$= \sqrt{r^2 + {\cdot}49p^2}$ ohms

P.D. across coil terminals $\bar{\mathrm{V}}_1 = \bar{\mathrm{C}}\mathrm{I}' = \bar{\mathrm{C}}\sqrt{r^2 + {\cdot}49p^2}$ volts

P.D. across condenser terminals $\bar{\mathrm{V}}_2 = \dfrac{1}{Kp}\bar{\mathrm{C}} = \dfrac{10^5}{p}\bar{\mathrm{C}}$ volts

If we take a frequency n of 10 ~ per sec., then $p = 62{\cdot}8$. Hence we have—

Resistance $r = 30$ ohms

Inductive reactance $Lp = {\cdot}7p = 43{\cdot}9$ ohms

Capacity reactance $= \dfrac{10^6}{Kp} = \dfrac{10^5}{p} = 1595$ ohms

Total reactance $x = Lp - \dfrac{1}{Kp} = 43{\cdot}9 - 1595$

$= -1551$ ohms

Total impedance $\mathrm{I} = \sqrt{r^2 + x^2} = \sqrt{30^2 + 1551^2}$

$= 1551{\cdot}3$ ohms

Current $\bar{\mathrm{C}} = \dfrac{\bar{\mathrm{V}}}{\mathrm{I}} = \dfrac{200}{1551{\cdot}3} = {\cdot}1289$ amperes

Angle of lag $\tan\phi = \dfrac{x}{r} = -\dfrac{1551}{30} = -51{\cdot}7$

$\phi = 88{\cdot}85°$ leading

Impedance of coil only $I_1 = \sqrt{r^2 + L^2p^2}$

$= \sqrt{30^2 + 43{\cdot}9^2} = 53{\cdot}15$ ohms

P.D. across coil terminals $\bar{V}_1 = I'\bar{C} = 53{\cdot}15 \times {\cdot}1289$

$= 6{\cdot}85$ volts

P.D. across condenser terminals $\bar{V}_2 = \dfrac{1}{Kp}\bar{C} = 1595 \times {\cdot}1289$

$= 205{\cdot}7$ volts

A curious thing will be noted about this result even at this low frequency, viz. that the P.D. across the condenser terminals is 205·7 volts, which is higher than the supply voltage. Further, the sum of the voltages across the coil and condenser added together, viz. 205·7 + 6·85, is no less than 212·6 volts. This appears absurd in the light of direct-current working, where the sum of the voltages dropped over each portion of the circuit *must* equal the supply voltage; but, as we shall see later, the present result is quite correct and rational for alternate currents.

A table for various frequencies can then be drawn up, as on p. 136.

The table and the curves (Fig. 83) show the remarkable changes which occur, as the frequency is raised, while the main voltage remains constant. The inductive reactance Lp increases in proportion to the frequency as shown by the sloping straight line, while the capacity reactance $\frac{1}{Kp}$, which is reckoned negative, decreases. When the inductive and capacity reactances are equal, $Lp = \frac{1}{Kp}$, from which $p^2 = \frac{1}{KL}$ and the frequency $n = \frac{p}{2\pi} = \frac{1}{2\pi\sqrt{KL}}$. Since $K = 10$ mfds. $= 10^{-5}$ farads and $L = {\cdot}7$ henry, $n = \frac{1}{2\pi\sqrt{{\cdot}7 \times 10^{-5}}} = \frac{1000}{2\pi\sqrt{7}} = 60{\cdot}3$ periods. At this point $Lp - \frac{1}{Kp} = 0$, hence $I = \sqrt{r^2 + \left(Lp - \frac{1}{Kp}\right)^2} = r$ and the current $= \frac{V}{r} = \frac{200}{30}$ or 6·67 amperes. As the frequency is increased, therefore, the impedance drops to 30 ohms at 60·3 periods and the current increases to 6·67 amperes, after which the impedance increases, and the current falls as the

TABLE XIII.—CONDENSER AND CHOKING COIL IN SERIES

$K = 10$ mfds., $r = 30$ ohms, $L = ·7$ henry.

Frequency n	$p = 2\pi n$.	Inductive Reactance Lp Ohms.	Capacity Reactance $\frac{1}{Kp}$ Ohms.	Total Reactance $x = Lp - \frac{1}{Kp}$ Ohms.	Total Impedance $I = \sqrt{r^2 + x^2}$ Ohms.	Current $C = \frac{\overline{V}}{I}$ Amperes.	Tan $\phi = \frac{x}{r}$	Lag ϕ Degrees.	Impedance of Coil $I_1 = \sqrt{r^2 + L^2p^2}$ Ohms.	P.D. across Coil $\overline{V}_1 = I_1C$ Volts.	P.D. across Condenser $\overline{V}_2 = \frac{1}{Kp}C$ Volts.
0	0	0	∞	−∞	∞	0	−∞	−90	30	0	200
10	62·8	43·9	1595	−1551	1551·3	·129	−51·7	−88·85	53·2	6·86	206
20	125·6	87·8	797·5	−709·7	710·4	·28	−23·7	−87·59	92·8	26	223
30	188·4	131·9	531·6	−399·7	400·7	·499	−13·3	−85·7	135·2	67·6	265
40	251·2	175·6	398·8	−223·2	225·2	·889	−7·44	−82·35	178·1	158·3	354
50	314	219·5	318·5	−99·0	103·5	1·933	−3·30	−73·15	221·6	428	617
60	376·8	263·8	265·8	−2·0	30·1	6·65	−·0667	−3·8	265·5	1763	1766
70	439·6	307·7	227·9	+79·8	85·2	2·35	+2·66	+69·4	309·2	726	535
80	502·4	351·2	199·4	151·8	154·8	1·292	5·06	78·8	352·5	456	258
90	565·2	395·1	177·2	217·9	219·9	·91	7·60	82·5	396·2	360	161·2
100	628	439	159·5	279·5	281·2	·712	9·32	83·85	440	313	113·5
150	942	658·5	106·2	552·3	553·1	·362	18·41	86·9	659·2	238·3	38·5
200	1256	878	79·75	798·3	798·9	·250	26·6	87·85	878·5	219·6	20
300	1884	1319	53·16	1266	1266·4	·158	42·2	88·65	1319·4	208·1	8·4
∞	∞	∞	0	∞	∞	0	∞	90	∞	200	0

frequency rises. The sharpness of the rise and fall of the current is clearly shown in the diagram. But the remarkable feature of the table is the variation of the voltages at the

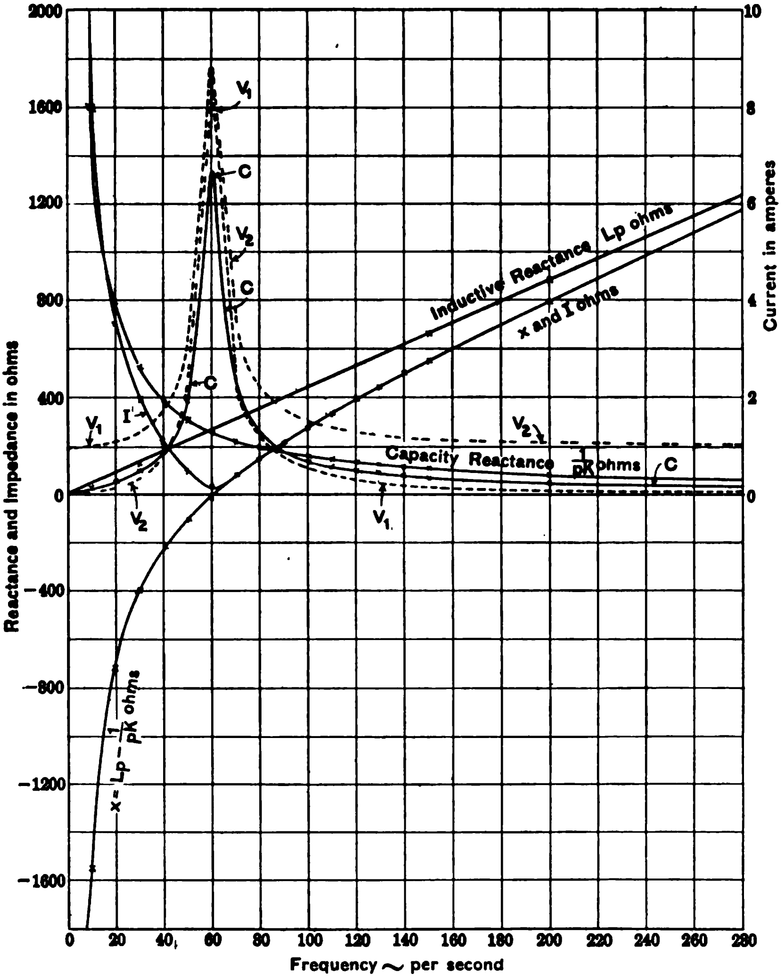

FIG. 83.—Variation of current, &c., in condenser and choking coil in series.

terminals of the coil and condenser. It has already been mentioned that at a low frequency the voltage across the condenser exceeds that of the mains, but as the frequency rises this effect is increased, until at a frequency of 60 periods the P.D. across the coil is no less than 1763 volts, and that across the condenser 1766 volts, although the main voltage is only 200. On increasing the frequency these voltages both drop, but that across the coil always remains greater than 200 volts. The explanation of these enormous voltages is given in the vector diagram (Fig. 84), from which it is seen that the two voltages V_1 across the condenser and that across the coil, *when added as vectors, do* make the 200 volts on the mains.

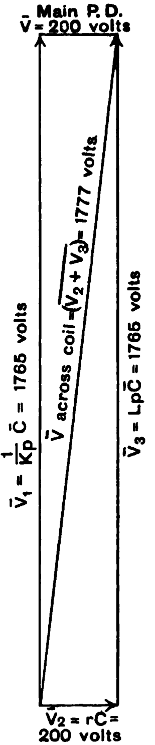
Main P.D.
$\bar{V} = 200$ volts
$\bar{V}_1 = \frac{1}{Kp}\bar{C} = 1765$ volts
$\bar{V}$ across coil $= \overline{(V_2 + V_3)} = 1777$ volts
$\bar{V}_3 = Lp\bar{C} = 1765$ volts
$\bar{V}_2 = r\bar{C} =$
200 volts

FIG. 84.—Vector triangle of voltages for coil and condenser in resonance.

This effect appears most extraordinary and incredible, but is unfortunately only too true in some practical cases. To realise it thoroughly we have only to go to our mechanical analogies, and there are plenty of them which are familiar to every engineer.

First take an ordinary pendulum, which has a gravitational pull towards the centre like a spring, and a bob of definite mass, and give it a very slight tap. This will start it swinging, but only through a short distance. Now repeat the tap at regular intervals with about the same intensity each time. If there is no particular relation between the time of swing of the pendulum and the time between the taps, the impulse will be given sometimes when the pendulum is swinging in the first direction and sometimes when it is on the back swing. The taps will therefore sometimes increase the swing of the pendulum and sometimes tend to stop it, and the average result will only be a very small swing. But now suppose that the speed of the taps is altered until they exactly agree with the time of swing of the pendulum. In that case every tap meets the pendulum when it is moving in the same direction, and consequently increases its swing; and after a short time the pendulum will be swinging over a distance perhaps ten times greater, or even much more than ten times that caused by the slower taps of exactly the same force. This means that with the same force or 'voltage' the velocity of the pendulum or 'current' is greatly increased when the frequency of the force is the same as the natural frequency of the pendulum. If the speed of the taps is further increased, they will again sometimes help and sometimes retard the pendulum, and the resultant swing will be smaller.

This accounts perfectly well for the rise of the current and its subsequent fall when the frequency is increased, but what about the extraordinary rise of voltage? Well, precisely the same thing takes place in the pendulum. The force of gravitation on the pendulum bob tending to pull it back to its lowest point is the same whether the pendulum is held stationary at that point or whether it has swung up to it. Now the force of the single tap was only sufficient to move the pendulum through a very small angle, and yet a number of similar taps properly timed produces a large swing, which means that a force is exerted on the bob many times greater than that of the actual impulse, just as the voltage across the condenser was much higher than the applied voltage of the mains. What is the explanation of this? After the pendulum has been set swinging, consider what happens if we cease driving it or touching it at all. The motion does not immediately cease; in fact, if the pendulum is well pivoted, it will go on swinging backwards and forwards for an hour or more. Here, therefore, we have large forces expended in accelerating the mass of the bob, and in lifting it against the elastic force of gravitation, without any applied force at all. It is evident that what really takes place is that, when the pendulum is at the end of its swing, the force of the earth's pull on it accelerates the mass, and this acceleration, although less and less as the bob descends, continues until it reaches the centre of the swing. By this time the bob has gained a considerable momentum, and when it has reached the centre or lowest point and the earth no longer pulls it forwards, the inertia or momentum carries it forwards even against the earth's pull, which is now backwards towards the centre. This gives the pendulum a negative acceleration which, however, does not stop it until it has reached practically the same height on the other side, after which it again swings back in the opposite direction. In other words, the inertia produces the force to overcome gravitation, and gravitation the force to overcome inertia, and the interaction of these two would cause the oscillation to go on for ever without any external force, were it not for the small amount of air or mechanical friction which must be compensated for. Anyone who has watched an ordinary grandfather clock with its long, heavy pendulum must have realised that the minute force given by the pallets of the escapement is almost infinitesimal in comparison with the force required to move the pendulum to the end of its swing. This exactly agrees with what we found in our electrical calculation, for when $Lp = \frac{1}{Kp}$, $\bar{C} = \frac{\bar{V}}{r}$ or the current is exactly that which would flow with the resistance or electrical friction only, and the capacity and inductance each supply the necessary force to the other, to keep up the oscillation. At lower frequencies than this the inertia effect is smaller, and consequently the capacity has the upper hand. The current should therefore be smaller and leading, as is shown in the curves, while at higher frequencies the inductance or inertia predominates, choking down the current and causing it to lag.

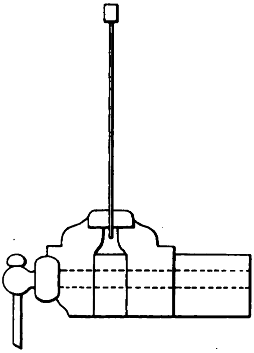

FIG. 85.—Spring and weight clamped in vice.

An even clearer example of this is the combination of a spring and a weight, as in Fig. 85. If a steel bar with a weight on the end of it is clamped in a vice, and is pulled to one side and let go, it will vibrate to and fro with a definite frequency. If we take hold of the end of it and move it backwards and forwards very slowly, we shall feel the elastic force of the spring, which is greatest at the end of the stroke, practically as if there were no weight. But now start to move it more rapidly, and it will be found that it swings much more easily; or if the same force is exercised the swing will be much greater, until when we move it with the same frequency as that with which it vibrated itself, it will move almost without any force from the hand at all, and if we were to continue to exert the same force as at first, the swing would become so great as perhaps to break the spring (just as our condenser would probably have broken down with the 1763 volts). But if we try to move it much more quickly still, we shall now find a greater resistance to the motion owing to the inertia of the mass, and we shall also find that our greatest force in a given direction has to be exerted at the *beginning* of the stroke in that direction, whereas it had to be exerted at the *end* when we were moving slowly and the inertia effect was small in comparison with the elastic force of the spring. In other words, the force has changed from lagging to leading after passing the natural vibration frequency, just as with the electrical circuit containing capacity and inductance.

The phenomenon of great increase of motion for a given force is called *resonance* for a simple reason. If we go into a room in which there is a piano and sing a certain note, it will be found that the piano continues to give out that note for a short time after the voice has stopped. The piano contains a large number of strings, each of which when struck will vibrate with its own natural frequency and give out a certain note. When, however, an external sound is made the air vibrations produced by it strike against the piano strings, and act like a small alternating force. The particular string whose mass and elasticity cause it to have the same vibration frequency as the sound will therefore vibrate most by the principles which we have just explained, and after the exciting force or sound is stopped the string will continue to vibrate or *resound* just as the pendulum or the spring and weight continued to oscillate. Hence the term 'resonance' for the agreement of the frequency of the driving force or voltage with the natural oscillation frequency of the body or circuit.

We can easily see that the circuit consisting of the choking coil and condenser responds to a P.D. of its own natural vibration frequency in exactly the same way as the pendulum or piano responded to an impulse or vibration of its natural vibration frequency, by dealing with the capacity and inductance in the same way as we dealt with the spring and weight on p. 85. If we have a capacity of K farads in series with an inductance of L henries, then the P.D. at the terminals of the combination is $V = V_1 + V_3 = \frac{1}{K}Q + LA$. Now if the condenser is charged and then connected directly to the inductance, the applied voltage $V = 0$, and $\frac{1}{K}Q + LA = 0$, from which $A = -\frac{1}{KL}Q$. But if the quantity varied in a simple harmonic manner we have $A = -p^2 Q$, an equation of the same form as the last one. Hence the quantity actually must vary in a simple harmonic manner or oscillate with a frequency such that $p^2 = \frac{1}{KL}$ or $n = \frac{1}{2\pi\sqrt{KL}}$. We shall deal with these oscillations more fully later, but what we are concerned with at the moment is that this frequency, which is the natural frequency of electrical oscillation of the combination of capacity and inductance, is exactly the same as that for resonance when $Lp = \frac{1}{Kp}$, or $p^2 = \frac{1}{KL}$ as above.

A pretty and useful illustration of this principle of resonance is given in the frequency or speed indicators of Hartmann and Kemf, and of Frahms. In these instruments a number of timed reeds, with little white flags on the end of them, are fixed side by side like the vibrating reeds of a musical box, and the vibrations due to the rotation of a shaft or an alternating current are communicated to them either mechanically or magnetically. The reeds are each one adjusted to vibrate naturally at a certain rate, say 40, 41, 42, &c., up to 70 ~ per sec. If, now, such an instrument is vibrated by an engine or alternate current having a frequency of, say, 50 ~ per sec., only the reed which naturally vibrates at this rate will get up a large swing. Those near to this rate, say 49 and 51, will also vibrate to a fair extent, but those below and above these rates very little. Fig. 86 shows a photograph of one of these frequency indicators under these conditions, and it will be seen that the amplitude of the swing of the successive reeds rises and falls almost exactly like the diagram in Fig. 83 for the capacity-inductance combination, as our analogies should indicate.

A striking instance of the large force which can be set up by properly timed impulses is given by a strong but elastic structure, such as a suspension bridge. A bridge of this kind may be strong enough to support a dead or uniformly moving load of many tons, but if a single person stands at the centre of the span, and jumps in such a manner that his impulses are timed with the natural vibrations of the bridge, he may get up an oscillation greater than would be produced by its maximum load, and strain the structure. Anyone who has experienced the crossing of a ditch by an elastic plank, or the motion of a diving spring-board, will appreciate this. For this reason it is common in marching a squad of men over a suspension bridge to order them to break step, as otherwise their steps might happen to be in time with the oscillations of the bridge.

The phenomena of resonance can of course be readily shown with the mechanical model circuit as described on p. 109. If the piston is moved forwards and backwards so as to produce an alternating force, we find that when we have either the spring alone, or a weight only on the truck, the motion is small at ordinary frequencies. But when the spring and weight are both in use and the frequency of the movement of the piston is varied without altering the force, as will be the case if the stroke of the piston is kept constant, we find that at one speed the car oscillates to and fro with an amplitude many times greater than that with the spring or weight only, and the spring is evidently considerably strained. If the frequency is increased beyond this the oscillations rapidly die down again, and the change of phase is obvious.

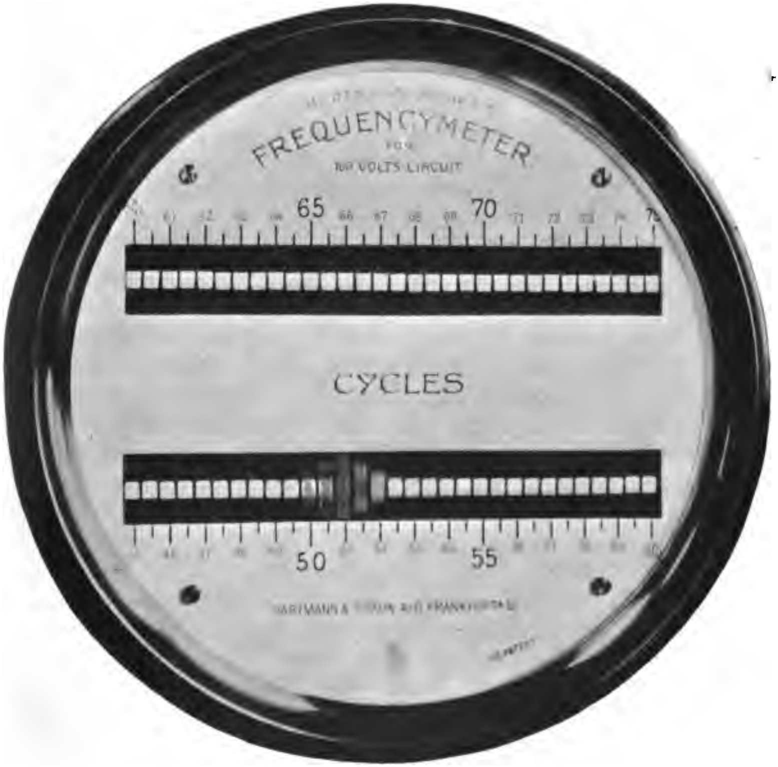

FIG. 86.—Frequency indicator.

These examples should more than suffice to make the phenomena of resonance seem perfectly simple and natural, and to dispel any idea of paradox. The importance of a thorough understanding of electrical resonance phenomena cannot be overrated, as they are frequently the cause of very disconcerting variations of pressure and of breakdowns in long-distance alternate-current distribution. The alternators, transformers, &c., used for the supply always have some inductance, and concentric or other cables have always a capacity of the order of a quarter or half a microfarad per mile. It may therefore be said that whenever a generator or transformer is running on unloaded cables, we have a capacity and inductance in series, and since the resistance of the coils of the machine and the cable conductors is usually very low, the current and rise of pressure, should resonance be set up, would be extremely large. Fortunately, under most practical conditions the frequency of the supply is considerably below the oscillation frequency of the circuit, and we do not therefore get complete resonance, but, as will be seen later, most machines give more or less irregular wave forms, and these irregularities may produce resonance and rise of pressure, though to a less degree. The subject is so important that it will be more fully dealt with in another volume.

Application to the Vibration of Machinery and Foundations.—In practically all reciprocating machines the motion of the parts set up alternating forces which tend to make the whole machine vibrate; while even revolving machines, such as electromotors, unless very perfectly balanced, do the same thing. The vibration so caused may frequently be a great source of annoyance, and very costly devices have often been employed to reduce it. It will be seen that the phenomena are precisely those which we have been studying. The reciprocating force set up by the motion of the machine corresponds to an alternating voltage, while the machine and its foundations have always a certain amount of elasticity corresponding to capacity, and of mass or inertia corresponding to inductance. There is also generally a certain amount of friction or resistance, due to the rubbing together of the particles of the concrete or other material forming the foundations, if there is any vibration. The machine with its supports is thus equivalent to a combination of capacity, resistance, and inductance, and the amplitude of its vibration under the action of the alternating force corresponds to the electrical quantity or is proportional to the current at a given frequency.

According to what we have now learnt, therefore, for every machine and foundation there ought to be a particular frequency or speed at which resonance will take place and the vibration show a marked increase. That this is the case must be well known to every observant user of machinery. If an engine or motor is started from rest and gradually run up to normal or above normal speed, it will almost always show a remarkable amount of vibration or knocking at one particular speed, although this vibration may be quite inappreciable at lower or higher speeds than that referred to. If even a well-balanced electromotor is fixed on the upper floors of an iron-girder building and is run up to speed, it will frequently cause a considerable vibration of the whole floor at a certain critical speed, although at normal speed we may not be conscious of its existence.

It will now be clear why machines often behave quite differently when fixed in different positions, and why often an almost absurdly small change of the mode of fixing will produce much greater or less vibration. The frequency at which resonance occurs depends upon the 'capacity' and 'inductance,' or the elasticity and inertia of the machine or foundations. Should it happen that this elasticity and inertia are just such as to give a natural vibration frequency near to the normal speed of the machine, there will always be serious vibration when running, and if the elasticity and inertia of any buildings in the neighbourhood correspond they will be seriously affected.

In the ordinary way it would generally be supposed that the remedy for excessive vibration would be more rigid foundations, but our alternate-current example shows that there are three quantities which can be varied—elasticity or rigidity, friction, and mass. The variation of the elasticity and inertia alters the frequency at which resonance will take place, so that either stiffening or lightening the foundations may produce the desired result. The simplest way of seeing whether the vibration is due to resonance or not is to start the machine and gradually run it up to speed. If the vibration only increases steadily with the speed, then the trouble is not due to resonance, and the oscillation must be diminished either by larger foundations or by damping them by asphalte or other viscous substances which are equivalent to electrical resistance, and thus reduce the current or oscillation. But if on raising the speed the oscillation suddenly develops on approaching the working speed it is due to resonance, and the mere tightening or loosening of a bolt, thus altering the elasticity, will sometimes reduce the vibration considerably. Whatever affects the rigidity or mass of the foundations will then diminish the vibration, by increasing or diminishing the rigidity or the inertia. Of course, the only method of stopping vibration at every possible speed is to have infinite rigidity or mass, which means $Y = 0$, or f or $m = \infty$;

the equivalents of these are a condenser of zero capacity or a dead break in an electrical circuit.

The thoughtful student will find many other rough analogies or resemblances to the phenomena we have dealt with. He will notice how persons of ability or capacity tend to lead, while those who are sluggish or who possess what Faraday called mental inertia tend to lag, like inductive circuits. But it is not necessarily the greatest capacity or lead which is most successful, as it is too much out of phase with the times, but rather the combination of brilliancy with stability, or of capacity and inductance, which is in resonance or harmony with the rate of progress and in phase with it, while a very little excess of either quality reduces the amplitude considerably. Similarly, in society as a whole we have the radical or leading element, and the conservative or inductive or lagging element which, when combined, are very prone to produce violent oscillations, like the spring and weight, unless damped out by the intermediate class which is more or less in phase with a suitable rate of progress, and thus may be thought of as a non-inductive resistance. These illustrations may be thought irrelevant and childish, but are really of some importance, as they illustrate that tendency to 'hunting' or oscillation which always seems to accompany all movement or evolution whether geological, biological, or social, and also, perhaps, show that each characteristic has its useful as well as its objectionable features.

Compound Impedances in Series.—Since the effective voltages in the case of circuits in series can be added as vectors, as shown on p. 125, it follows that in the case of any number of coils or circuits each having resistance and reactance the impedances which are the effective voltages for unit effective current can be added like vectors to get the total impedance in magnitude and phase. Or we have the following rule:—

The joint impedance of any number of circuits in series is obtained by adding the separate impedances as forces or vectors.

CHAPTER XI

COMBINATIONS OF RESISTANCE, INDUCTANCE, AND CAPACITY IN PARALLEL CIRCUITS

THE calculation of the currents in any combination of resistance, inductance, and capacity in series should now present no difficulties. We have still to deal with such combinations when joined in parallel.

Before commencing to attack the problem for alternate currents, it will be well if we first run over the mode of dealing with direct-current circuits in parallel, as the alternate-current problem will follow it closely. In Fig. 87 we have a combination of three resistance coils in parallel, the resistance of each of them being respectively r_a, r_b, and r_c ohms. If we imagine that a P.D. of V volts is applied to the terminals of the three, then we have a current $C_a = \frac{V}{r_a}$ in the first, of $C_b = \frac{V}{r_b}$ in the second, and of $C_c = \frac{V}{r_c}$ in the third. But according to Kirchhoff's first law the algebraic sum of the currents at the junction of the resistances is zero, hence the total main current

$$C = C_a + C_b + C_c = \frac{V}{r_a} + \frac{V}{r_b} + \frac{V}{r_c} = V\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right)$$

Now if the whole of the three resistances had been replaced by a single one of resistance r ohms such that the current $\frac{V}{r}$ was the same as the above main current, then

$$\frac{V}{r} = V\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \quad \text{or} \quad \frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \quad . \quad (29)$$

where r is the resistance which is equivalent to or gives the same current for the same P.D. as the combination gives. This evidently applies to any number of circuits, and can be expressed by saying that when any number of circuits are in parallel, the reciprocal of the joint resistance is equal to the sum of the reciprocals of the component resistances. But this immediately suggests that we should coin a word for the reciprocal of a resistance, so that the working out of any number of circuits in parallel should only be a matter of addition. Now the conducting power of a substance is inversely proportional to its resistance, since the better a conductor the lower its resistance, and we might justly speak of the reciprocal of the resistance of a conductor as its conducting power. Lord Kelvin employed the term *conductivity* in *mhos* ('ohm' spelt backwards to denote the reciprocal), and we could then say that the conductivity of a circuit in mhos = unity divided by its resistance in ohms, and that the joint conductivity of any number of circuits in parallel is the sum of the separate conductivities of the various branches.

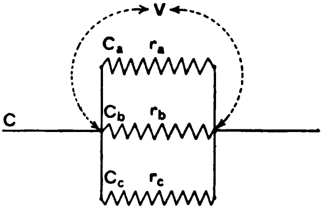

FIG. 87.—Resistances in parallel.

The term 'conductivity' has, however, not been considered quite suitable, as it does not harmonise with other words, such as 'resistance,' 'reactance,' 'inductance,' &c., and, further, the termination *ity* or *ivity* generally implies a *specific* property of a substance, as, for example, specific gravity, density, permeability, heat-conductivity, &c.

Prof. Steinmetz has therefore proposed the term *conductance* in mhos for the reciprocal of the resistance in ohms, and denotes it by the letter g, so that for any resistance $g_{\text{mhos}} = \dfrac{1}{r_{\text{ohms}}}$, and for any number of resistances in parallel the joint conductance $g = g_a + g_b + g_c$, or the joint conductance is the sum of the separate conductances. This nomenclature has been now generally adopted.

Alternate-Current Circuits in Parallel: Kirchhoff's First Law for Alternate Currents.—Whenever any number of conductors meet in a point, then Kirchhoff's first law, which, as we have pointed out, simply expresses the incompressibility of the 'electric fluid,' states that the algebraic sum of the currents flowing towards and away from that point is zero. Now this is true not only for steady currents but for unsteady ones, provided that we take the sum of the currents in all the branches leading to any junction point at the same instant. For if it were not true at each instant, there would be a short time during which electricity is accumulated at or withdrawn from the junction, and this is certainly impossible in any ordinarily measurable small time, although we are not perhaps justified in assuming it absolutely impossible if we were dealing with millionths of a second. For all ordinary conditions, however, it may be assumed that the algebraic sum of the currents is zero at every instant, and consequently that, when we have a main conductor supplying several branches, the main current at every instant is equal to the sum of the currents in the branches. Now suppose that each of the currents in the branches varies in a simple harmonic manner. Then the value of the main current is got by adding together the values of the branch currents from instant to instant, or is the sum of the simple harmonic currents in the branches. But we have already learnt in the last chapter that the sum of any number of simple harmonic P.D.'s in series is also a simple harmonic P.D., the effective value of which is obtained by adding together the effective values of each of the component P.D.'s as cranks or vectors. The same thing must therefore hold for the *currents* when dealing with circuits in parallel, and we therefore have the following rules:

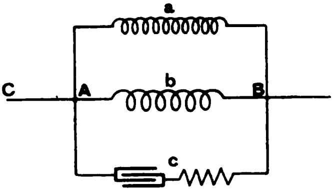

FIG. 88.—A.C. circuits in parallel.

Whenever a number of conductors meet in a point and are traversed by sinusoidal alternating currents of the same frequency, the vector sum of their effective values is zero.

Whenever a main conductor supplies sinusoidal alternating currents to any number of branches, the effective main current is the vector sum or resultant of the effective values of the currents in the branches.

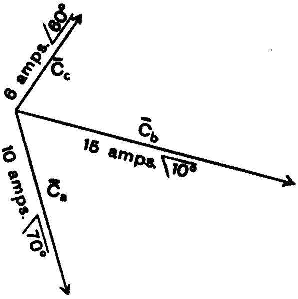

FIG. 89.—Alternate-current vectors.

For example, suppose we have three circuits between the points A and B (Fig. 88), one (a) very inductive in which the current C_a is, say, 10 amperes lagging 70°, another (b) slightly inductive taking a current $C_b = 15$ amperes lagging 10°, and a third (c) containing a condenser and having a current of $C_c = 6$ amperes leading 60°. Then by representing these three currents in vector form, as in Fig. 89, and adding them together,

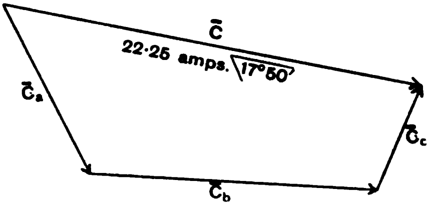

FIG. 90.—Resultant current.

as in Fig. 90, we obtain the total current in the mains as $22{\cdot}25\,\overline{\diagdown}\,17^\circ\,50'$ effective amperes.

It is therefore obvious that we can find the total current in the mains whenever we have circuits of given impedances in parallel. Take the simple case of a coil having a resistance of 50ω and an inductance of ·75 henry, in parallel with a 15-microfarad condenser and a series resistance of 100 ohms, as in Fig. 91, on 200 volt 50 ~ mains. Here we have $p = 2\pi n = 314$, and the inductive reactance $Lp = {\cdot}75 \times 314 = 235{\cdot}5\omega$, while the capacity reactance $-\dfrac{1}{Kp} = -\dfrac{10^6}{15 \times 314} = -212{\cdot}5\omega$. Consequently the impedance of the inductive coil $\mathrm{I}_a = \sqrt{r^2 + x^2} = \sqrt{50^2 + 235{\cdot}5^2} = 240\omega$, and $\tan\omega = \dfrac{235{\cdot}5}{50} = 4{\cdot}71$ or $\phi = 78^\circ$.

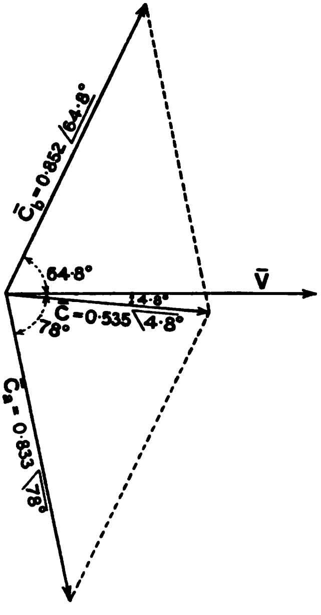
$C_b = 0{\cdot}852\,\angle 64{\cdot}8^\circ$
64·8°
$\bar{V}$
14·8°
$\bar{C} = 0{\cdot}535\,\diagdown 4{\cdot}8^\circ$
78°
$C_a = 0{\cdot}833\,\angle 78^\circ$

FIG. 91.—Resultant current for choking coil and condenser.

Similarly, the impedance of the condenser circuit $\mathrm{I}_b = \sqrt{r^2 + \left(\dfrac{1}{Kp}\right)^2} = \sqrt{100^2 + 212{\cdot}5^2} = 234{\cdot}8\omega$, and

$\tan\phi = \dfrac{212{\cdot}5}{100} = 2{\cdot}125$, from which $\phi = 64{\cdot}8^\circ$. Hence on 200 volts we have $\bar{C}_a = \dfrac{\bar{V}}{I_a} = {\cdot}833 \mathrel{\overline{\diagdown}} 78^\circ$ amperes, and $\bar{C}_b = \dfrac{\bar{V}}{I_b} = {\cdot}852 \angle 64{\cdot}8^\circ$ amperes. Combining these two graphically in Fig. 91, we find the sum or resultant as $\cdot535 \mathrel{\overline{\diagdown}} 4{\cdot}8^\circ$ amperes.

But this process requires the use of drawing instruments. Can we not obtain a general formula as we did for circuits in series?

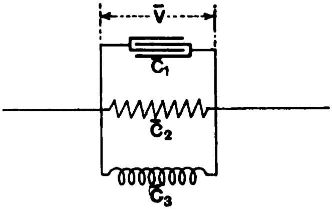

FIG. 92.—Condenser, resistance, and inductance in parallel.

If we have circuits containing purely capacity, resistance, and inductance in parallel, a formula is obtained quite easily. In Fig. 92 we have a condenser, a resistance, and an inductive coil of negligible resistance across mains having an effective voltage $\bar{V}$. Then the condenser current $\bar{C}_1 = Kp\bar{V} \angle 90^\circ$, the current in the resistance $\bar{C}_2 = \dfrac{\bar{V}}{r}$, and that in the inductance is $\bar{C}_3 = \dfrac{\bar{V}}{Lp} \mathrel{\overline{\diagdown}} 90^\circ$. We can combine these, as in Fig. 93, and we get a right-angled triangle, of base $\bar{C}_2 = \dfrac{\bar{V}}{r}$ and vertical side $\bar{C}_3 - \bar{C}_1 = \bar{V}\left(\dfrac{1}{Lp} - Kp\right)$, from which the total current

$$\bar{C} = \sqrt{\bar{C}_2^2 + (\bar{C}_3 - \bar{C}_1)^2} = \bar{V}\sqrt{\frac{1}{r^2} + \left(\frac{1}{Lp} - Kp\right)^2} \qquad (30)$$

while the tangent of the angle of lag or

$$\tan\phi = \frac{\bar{C}_3 - \bar{C}_1}{\bar{C}_2} = \frac{\dfrac{1}{Lp} - Kp}{\dfrac{1}{r}} = r\left(\frac{1}{Lp} - Kp\right) \qquad (31)$$

An interesting consequence follows at once from this result. Suppose that the inductance, capacity, and frequency are such that $\frac{1}{Lp} = Kp$, or that $Lp = \frac{1}{Kp}$, which is the same condition as we had for resonance with circuits in series. Then $\frac{1}{Lp} - Kp = 0$ and $\bar{C} = \frac{\bar{V}}{r}$, while $\tan\phi = r\left(\frac{1}{Lp} - Kp\right) = 0$, from which $\phi = 0$. Hence the current is simply, in magnitude and phase, that which would be produced in the resistance r alone. This is exactly what we found for resistance, inductance, and capacity in series in the case of resonance. But now suppose that we have a condenser, and inductance in parallel, without any resistance. Then $\bar{C}_1 = Kp\bar{V} \angle 90^\circ$ and $\bar{C}_3 = \frac{1}{Lp}\bar{V} \overline{\diagdown} 90^\circ$ and $\bar{C} = \bar{C}_1 + \bar{C}_3 = \left(\frac{1}{Lp} - Kp\right)\bar{V} \overline{\diagdown} 90^\circ$ simply. Hence if $Lp = \frac{1}{Kp}$, $\bar{C} = 0$, or there is no current in the mains, although the condenser and inductive coils are each taking their full currents. This is another of the paradoxes of alternate currents which seem so mysterious and absurd. But the explanation is very simple when the problem is looked at from the mechanical point of view. We have only to consider how a spring and a weight without friction can be put mechanically 'in parallel.' Now whenever a cord passes over a pulley, the tension or pull in the cord on both sides of the pulley is the same. Hence if, as in Fig. 94, a pulley with a cord attached on one side to a spring and on the other side to a weight, is held by the hand from the shackle of the pulley and made to oscillate up and down, we always communicate an equal force to both sides of the cord, as an equal P.D. is communicated to each of two circuits in parallel. It is true that the force which we apply to the shackle is the sum of the tensions in the cords, and therefore double of each of them instead of being equal to them, as is apparently the case in the electric circuit, but this is only because we are comparing a total force, in the case of the cords, with a pressure or force per unit area in the electrical case. We must therefore take half the force applied to the shackle, or the force per cord, as the P.D. in the mechanical example. A more perfect example

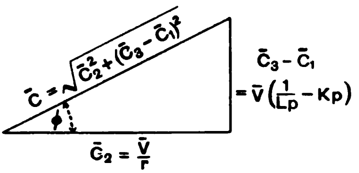
$\bar{C} = \sqrt{\bar{C}_2^2 + (\bar{C}_3 - \bar{C}_1)^2}$
$\bar{C}_3 - \bar{C}_1$
$= \bar{V}\left(\frac{1}{Lp} - Kp\right)$
ϕ
$\bar{C}_2 = \frac{\bar{V}}{r}$

FIG. 93.—Vector diagram for condenser, resistance, and inductance in parallel.

is shown in Fig. 95, where a water circuit is divided into two branches in parallel, one of which contains mercury so as to have considerable inertia or 'inductance,' while the other has a tank with elastic partition or liquid condenser, as in Fig. 16. In either of these cases, if a sudden force is applied to the shackle or water, the inertia prevents motion in one branch at first, while the spring or partition yields. After a short interval, however, the stretching of the spring causes a tension in the cord which accelerates the mass, and when the mass has got up speed it will tend to run on and to let the spring recoil. Now it is clear that if a motion of this kind has been started and the shackle of the pulley is held stationary, or the main water pipe is shut off, there will still be an oscillating motion of the cord or water round the pulley or parallel water circuit. In fact, if the outside motion is stopped, we have simply the spring and weight or the condenser and inductance in series with one another round the loop, and therefore there will be a natural time of oscillation for which $p^2 = \frac{1}{Ym}$ or $\frac{1}{KL}$ exactly as in an ordinary series circuit. If there were no friction this oscillation

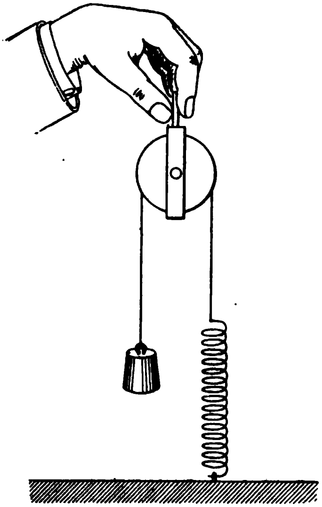

FIG. 94.—Spring and weight in parallel.

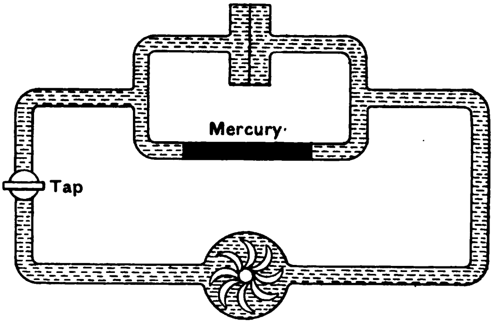

FIG. 95.—Liquid circuits in parallel.

would go on continuously without dying down, and we then have motion of both the inertia and spring in the two branches, without any external motion of the pulley shackle or water in the main pipe. But this is exactly the equivalent of two currents flowing in two circuits in parallel, without any current in the mains. If the main pipe is opened or the shackle is released, an alternating force must be provided in order to balance the forces set up by the motion of the spring and weight or of the water, and if the shackle is held in the hand while the cord is oscillating, these forces will be distinctly felt. It is therefore necessary to apply an alternating P.D. to a parallel combination of inductance and capacity, but if this force is in unison with the natural vibration frequency of the circuit for which $Lp = \frac{1}{Kp}$, there will be no current in the mains, except in so far as the inductance and condenser are not perfect, and require a small amount of current from the supply in order to make up for the tendency of the oscillations to die away.

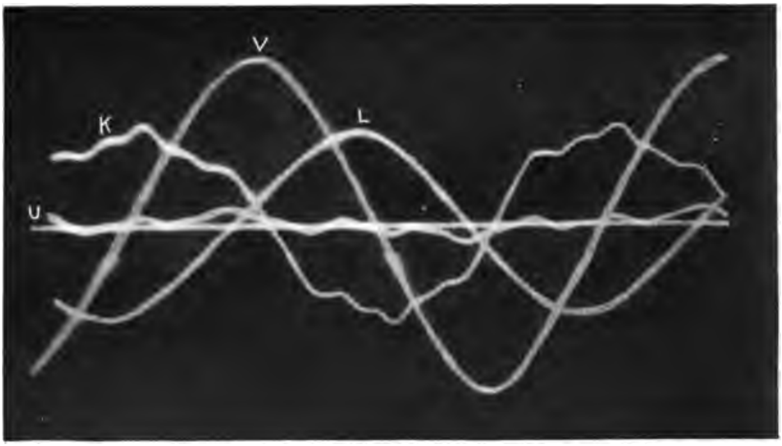

FIG. 96.—Oscillograph curves from condenser and inductance in parallel.

Fig. 96 shows oscillograph curves for a combination of capacity and inductance in parallel. Curve *V* is of the supply voltage, *K* of the current when the condenser is alone connected, *L* of the current for the inductance only, and *U* the resultant main current when both are connected in parallel. It will be seen that the condenser current shows slight irregularities, as a capacity or spring responds very readily to irregularities in the applied P.D.

or force, while inductance or inertia tends to smooth them out. When placed in parallel, therefore, the inductance and capacity can only compensate for one another as far as the principal current wave is concerned, and the mains supply the irregularities, as shown. With a perfectly sinusoidal P.D. the resultant current at the right frequency would be practically nil.

This property of condensers and choking coils to mutually supply current to each other, or to compensate for one another, is often of great value in practice. As a rule an alternate-current supply system is considerably inductive owing to the number of transformers, motors, and arc lamps in circuit, all of which act more or less like choking coils. This, of course, means that the main current will lag behind the E.M.F. of the generators; and, as will be seen later, this tends to bad regulation of the voltage of the circuit. But, as we know, long cables act as condensers, and consequently tend to compensate for this inductance, and further, as we shall also see later, synchronous motors can be made to act like condensers, and are often so used to correct the inductive effect of other machinery. It is possible yet that with improved manufacture of condensers they may be used extensively on alternate-current supply systems.

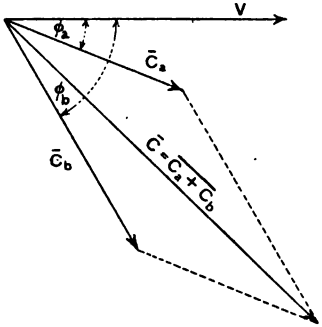

FIG. 97.—Composition of currents in parallel.

Calculation of Parallel Circuits in General.—The formula obtained for capacity, resistance, and inductance in parallel, though interesting, is not sufficiently general, as practical circuits do not as a rule fulfil the conditions laid down. It is true that a condenser, and a non-inductive resistance, may for most purposes be taken as a pure capacity and a pure resistance, but an inductive coil without resistance is unknown. This did not trouble us particularly when dealing with circuits in series, as the resistance of the inductive coil was simply part of the resistance of the circuit, and acted virtually as if it were in series with the inductance of the coil. But this will not do when we have the coil in parallel with other circuits. It is necessary for us, then, to consider the combination in parallel of circuits having both reactance and resistance mixed. To do this we have simply to find the currents in each of these circuits, and to add them together as vectors by calculation.

Suppose, therefore, that we have two circuits a and b, each having resistance and reactance, in parallel with one another. If r_a and x_a are respectively the resistance and reactance of the first circuit, and r_b and x_b those of the second circuit, then $\bar{C}_a = \frac{\bar{V}}{I_a} = \frac{\bar{V}}{\sqrt{r_a^2 + x_a^2}}$ and $\tan \phi_a = \frac{x_a}{r_a}$, from which we can show $\bar{C}_a$ as a vector in Fig. 97. Similarly, $\bar{C}_b = \frac{\bar{V}}{I_b} = \frac{\bar{V}}{\sqrt{r_b^2 + x_b^2}}$ and $\tan \phi_b = \frac{x_b}{r_b}$. In order to get the total current we have only to add $\bar{C}_a$ and $\bar{C}_b$ together as vectors (Fig. 97), and we shall have the total current $\bar{C}$, as obtained graphically before. But we want to be able to calculate the value of $\bar{C}$ without using instruments.

Now it is clear that the currents in both the branches are proportional to $\bar{V}$. Consequently, just as when dealing with circuits in series we found it convenient to deal with a current of one effective ampere, so we shall now find it convenient to find the currents in each circuit for one effective volt P.D. on the mains, to add these currents to get the total effective current for one volt, and afterwards to multiply the result by the actual voltage.

In the case of these two circuits we shall therefore have $\bar{C}_a = \frac{\bar{V}}{I_a} = \frac{1}{I_a}$ since $\bar{V} = 1$, and $\tan \phi_a = \frac{x_a}{r_a}$ as before. Similarly, $\bar{C}_b = \frac{\bar{V}}{I_b} = \frac{1}{I_b}$ and $\tan \phi_b = \frac{x_b}{r_b}$. Now it will be remembered that when we had combinations of resistance and reactance in series the number of volts P.D. between the ends of the circuit, when one effective ampere was passed through it, expressed the total impeding effect or the impedance of the circuit. When, however, circuits in parallel with steady currents were considered, then we had to add the currents, and we introduced the term 'conductance,' which was the current for one volt. The effective current for one effective volt in this case is $\frac{1}{I_a}$, but we are hardly justified in calling it the conductance of the circuit, as its magnitude is $\frac{1}{\sqrt{r^2 + x^2}}$ instead of $\frac{1}{r}$, and it is not in phase. For this reason a new term may be conveniently introduced, and Steinmetz has used the word *admittance* as representing the total admission of current to the circuit. It is generally denoted by the letter Y (which of course must not be confused with the yield-constant of a spring in the mechanical analogies), so that $Y_a = \frac{1}{I_a} = \frac{1}{\sqrt{r^2 + x^2}}$, and is reckoned like conductance in mhos.

Now we have seen that when any number of circuits are joined in parallel, the total current is the vector sum of the branch currents. As the admittances of the branches are simply the currents for one effective volt P.D., the same rule must hold, and we have the following important rule :

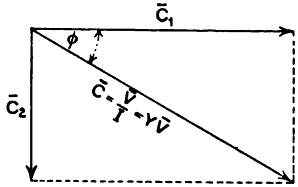

FIG. 98.—Energy and wattless components of current.

When any number of circuits are connected in parallel and are supplied with sinusoidal alternating currents, the joint admittance of the parallel circuits is the vector sum of the admittances of the branches.

This is the law for parallel circuits which corresponds with that given on p. 146 for series circuits, *i.e.* that the joint impedance was the vector sum of the separate impedances. But it does not yet help us to calculate the total admittance without graphical aid, as the separate admittances will not generally have the same phase, and cannot therefore be simply added together by their numerical magnitudes. There are various ways of calculating the resultant of a number of vectors, but one of the most suitable in this case is to split each vector into horizontal and vertical components, as is commonly done in mechanical problems and as we did for simple harmonic motions on p. 108. We can then add all the horizontal components together to get the horizontal component of the resultant, since they all have the same direction or phase; and similarly the vertical component of the resultant is obtained by adding together all the vertical components of the separate admittances, since *they* are all in phase. The total admittance is then given by taking the square root of the sum of the squares of the total horizontal and total vertical components, as they are at right angles.

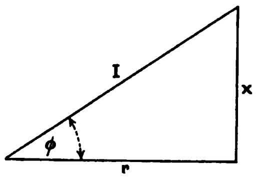

FIG. 99.—Impedance triangle.

Now what are the horizontal and vertical components of the admittance of a circuit? In Fig. 98 we have a current $\bar{C} = \frac{\bar{V}}{I} = Y\bar{V}$ lagging by the angle ϕ behind $\bar{V}$. By drawing lines parallel to and perpendicular to $\bar{V}$ we obtain two components $\bar{C}_1$ in phase with $\bar{V}$, and $\bar{C}_2$ lagging 90° behind it. It is easy to see that $\frac{\bar{C}_1}{\bar{C}} = \cos\phi$, or $\bar{C}_1 = \bar{C}\cos\phi = \frac{\bar{V}}{I}\cos\phi$, and similarly $\frac{\bar{C}_2}{\bar{C}} = \sin\phi$, or $\bar{C}_2 = \bar{C}\sin\phi = \frac{\bar{V}}{I}\sin\phi$. But if we consider the circuit having the impedance I, which is made up of the resistance r and the reactance x, we know from p. 125 that r, x, and I form a right-angled triangle, and that ϕ, the angle by which the current lags behind the P.D., is the angle between I and r in the triangle, Fig. 99. Hence we have immediately—

$$\cos\phi = \frac{r}{I} \quad \text{and} \quad \sin\phi = \frac{x}{I}$$

Putting these values in the above expressions—

$$\bar{C}_1 = \bar{C}\cos\phi = \frac{\bar{V}}{I}\cdot\frac{r}{I} = \frac{r}{I^2}\bar{V} = \frac{r}{r^2 + x^2}\bar{V} \quad . \quad . \quad . \quad (32)$$

$$\bar{C}_2 = \bar{C}\sin\phi = \frac{\bar{V}}{I}\,\frac{x}{I} = \frac{x}{I^2}\bar{V} = \frac{x}{r^2 + x^2}\bar{V} \quad . \quad . \quad . \quad (33)$$

which give us at once the two components of the current, in phase and in quadrature with $\bar{V}$ respectively. These two components, as will be seen later, are called the *energy* and the *wattless* currents.

If we now suppose the effective voltage $\bar{V} = 1$, then $C = \frac{\bar{V}}{I} = \frac{1}{I} = Y$, the admittance of the circuit. At the same time $\bar{C}_1 = \frac{r}{I^2}\bar{V} = \frac{r}{I^2}$ and $\bar{C}_2 = \frac{x}{I^2}\bar{V} = \frac{x}{I^2}$. These are, therefore, the two components of the admittance which we were seeking. What shall we call them? Now, if we had a circuit having resistance only, $\bar{C} = \frac{\bar{V}}{r}$ and is in phase. Hence the admittance $= \frac{1}{r}$ simply, which is the conductance g in mhos; the impedance $\mathrm{I} = \sqrt{r^2 + x^2} = r$, and $\bar{\mathrm{C}}_1 = \frac{r}{\mathrm{I}^2}\bar{\mathrm{V}} = \frac{r}{r^2}\bar{\mathrm{V}} = \frac{\bar{\mathrm{V}}}{r}$, while $\bar{\mathrm{C}}_2 = \frac{x}{\mathrm{I}^2}\bar{\mathrm{V}} = 0$, since there is no reactance. The admittance and its horizontal component are thus both equal to $\frac{1}{r}$, the conductance of the circuit, and the current is in phase with the P.D. But when the current is out of phase, Steinmetz has employed the term *conductance* for the component of the admittance in phase with the P.D., or $g = \frac{r}{\mathrm{I}^2}$. This does not clash with the previous definition $g = \frac{1}{r}$ for steady currents, since whenever the reactance is zero, as it is for steady currents, we have just seen that $\mathrm{I} = r$, and $g = \frac{r}{\mathrm{I}^2} = \frac{1}{r}$. In future, therefore, we shall define *the conductance of a circuit as the effective value of the component of the current in phase with the P.D. when the effective value of the P.D. is one volt; or as its resistance divided by the square of its impedance.* Or

$$g = \frac{r}{\mathrm{I}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

Next suppose that we have a circuit having reactance only, without resistance. Then $\mathrm{I} = \sqrt{r^2 + x^2} = x$ and $\frac{r}{\mathrm{I}^2} = 0$, while $\frac{x}{\mathrm{I}^2} = \frac{x}{x^2} = \frac{1}{x}$; also $\mathrm{Y} = \frac{1}{\mathrm{I}} = \frac{1}{x}$. Hence in this case the admittance and its vertical component are both $\frac{1}{x}$, and there is no component in phase, as we know. We cannot call this the conductance of the circuit, although it is the current for unit P.D., because the current is in quadrature instead of in phase with the P.D., just as we could not speak of x as a resistance because of the shift of phase. Consequently we again require a new name, and Steinmetz has given us the term *susceptance* for it and denotes it by b. *The susceptance of a circuit is the effective value of the component of the current lagging* 90° *behind the P.D. when the* *effective value of the P.D. is one volt; or its reactance divided by the square of its impedance.* Or

$$b = \frac{x}{\mathrm{I}^2} \quad . \; . \; . \; . \; . \; . \; . \quad (35)$$

The conductance and susceptance of a circuit are, therefore, the two components of its admittance, just as the resistance and reactance are the two components of its impedance. It follows that the admittance Y, being the hypothenuse of the right-angled triangle of which the conductance g and the susceptance b are the two perpendicular sides, is equal to the square root of the sum of the squares of the conductance and susceptance, or $\mathrm{Y} = \sqrt{g^2 + b^2}$, just as $\mathrm{I} = \sqrt{r^2 + x^2}$. This is also seen from the values $g = \frac{r}{\mathrm{I}^2}$ and $b = \frac{x}{\mathrm{I}^2}$, from which $g^2 + b^2 = \frac{r^2}{\mathrm{I}^4} + \frac{x^2}{\mathrm{I}^4} = \frac{r^2 + x^2}{\mathrm{I}^4} = \frac{\mathrm{I}^2}{\mathrm{I}^4} = \frac{1}{\mathrm{I}^2}$, and hence $\sqrt{g^2 + b^2} = \frac{1}{\mathrm{I}} = \mathrm{Y}$. Furthermore, the angle ϕ which is given by the equation $\tan\phi = \frac{x}{r}$, is also given by $\tan\phi = \frac{b}{g}$, as shown in the triangle (Fig. 100), or by taking the values for b and g, $\tan\phi = \frac{b}{g} = \frac{x}{\mathrm{I}^2} \div \frac{r}{\mathrm{I}^2} = \frac{x}{\mathrm{I}^2} \times \frac{\mathrm{I}^2}{r} = \frac{x}{r}$, the ordinary value.

These results may be summarised as follows:

TABLE XIV.—COMPARISON OF QUANTITIES FOR SERIES AND PARALLEL CIRCUITS

P.D. for Unit Effective Current.	Current for Unit Effective P.D.
Resistance r	Conductance $g = \frac{r}{\mathrm{I}^2} = \frac{r}{r^2+x^2}$
Reactance x	Susceptance $b = \frac{x}{\mathrm{I}^2} = \frac{x}{r^2+x^2}$
Impedance $\mathrm{I} = \sqrt{r^2+x^2}$	Admittance $\mathrm{Y} = \frac{1}{\mathrm{I}} = \sqrt{g^2+b^2}$
Lag:—$\tan\phi = \frac{x}{r}$	Lag:— $\tan\phi = \frac{b}{g}$

We must always remember that the conductance g and the susceptance b, expressed in mhos, besides being constants of the circuit (for a given frequency), are also the components of the current in phase with, and lagging 90° behind, the P.D. when one effective volt is applied, just as the resistance r and reactance x of a circuit are also the voltages respectively in phase with, and leading 90° in front of, a unit current passing through the circuit. Therefore, since all the currents in any number of circuits in parallel that are in phase with the P.D. are in phase with each other, their magnitudes or effective values may simply be added together, and the sum of their conductances gives the total current in phase with the unit P.D. of the mains, which is evidently the total conductance of the combination of circuits. Hence—

For any number of circuits in parallel the total or resultant conductance is the sum of the conductances of the separate circuits.

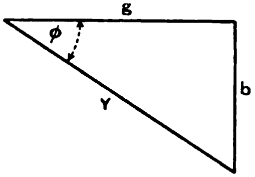

FIG. 100.—Conductance, susceptance, and admittance.

It will be seen in like manner that the susceptances, which are the currents lagging 90° behind the P.D. for one effective volt, are all in phase with one another, and their sum therefore gives the total quadrature current in the mains, which is the total susceptance. Hence—

For any number of circuits in parallel the total or resultant susceptance is the sum of the susceptances of the separate circuits.

If, therefore, we have any number of circuits in parallel, the resistance and reactance of each being given, we can find the conductance and susceptance of each branch, and, by simple addition, the conductance and susceptance of the combination. If we call this total conductance g and total susceptance b, then we have the total admittance, which is the total main current for a P.D. of one effective volt, $Y = \sqrt{g^2 + b^2}$, from which the total current for $\bar{V}$ effective volts will be $\bar{C} = Y\bar{V}$, lagging by an angle ϕ such that $\tan\phi = \frac{b}{g}$, and our whole problem is solved.

A numerical example should now make this perfectly clear, and we will take the case of the choking coil and condenser with series resistance which we worked out graphically before, on p. 150. Here we had $r_a = 50\omega$ and $x_a = Lp = {\cdot}75 \times 314 = 235{\cdot}5\omega$; and for the condenser circuit $r_b = 100\omega$ and $x_b = -\frac{10^6}{15 \times 314} = -212{\cdot}5\omega$.

Taking the choking coil first, we have $r_a = 50\omega$, $x_a = 235{\cdot}5\omega$, $I_a = \sqrt{50^2 + 235{\cdot}5^2} = 240\omega$; $\tan\phi_a = \frac{235{\cdot}5}{50} = 4{\cdot}71$. The conductance $g_a = \frac{r_a}{I_a^2} = \frac{50}{240^2} = {\cdot}000870$ mho, and the susceptance $b_a = \frac{x_a}{I_a^2} = \frac{235{\cdot}5}{240^2} = {\cdot}00409$ mho, while the admittance $Y_a = \frac{1}{I_a} = \frac{1}{240} = {\cdot}00417$ mho.

For the condenser circuit similarly we have $r_b = 100\omega$, $x_b = -212{\cdot}5\omega$, $I_b = \sqrt{100^2 + 212{\cdot}5^2} = 234{\cdot}8\omega$; $\tan\phi = \frac{212{\cdot}5}{100} = 2{\cdot}125$. The conductance $g_b = \frac{r_b}{I_b^2} = \frac{100}{234{\cdot}8^2} = {\cdot}001815$ mho, the susceptance $b_b = \frac{x_b}{I_b^2} = -\frac{212{\cdot}5}{234{\cdot}8^2} = -{\cdot}00385$ mho, and the admittance $Y_b = \frac{1}{I_b} = {\cdot}00426$ mho.

Combining these, for the two circuits in parallel we have—

$$\text{Total conductance } g = g_a + g_b = {\cdot}000870 + {\cdot}001815$$
$$= {\cdot}002685 \text{ mho.}$$

$$\text{Total susceptance } b = b_a + b_b = {\cdot}00409 - {\cdot}00385$$
$$= {\cdot}00024 \text{ mho.}$$

$$\text{Total admittance } Y = \sqrt{g^2 + b^2} = \sqrt{{\cdot}002685^2 + {\cdot}00024^2}$$
$$= {\cdot}00270 \text{ mho.}$$

$$\text{Resultant impedance } I = \frac{1}{Y} = \frac{1}{{\cdot}00270} = 371 \text{ ohms.}$$

$$\text{Lag :—} \tan\phi = \frac{b}{g} = \frac{{\cdot}00024}{{\cdot}002685} = {\cdot}0895, \text{ from which } \phi = 5{\cdot}1°.$$

For a P.D. of 200 volts we have $C = \frac{\bar{V}}{I}$ or $Y\bar{V} = {\cdot}00270 \times 200 = {\cdot}540$ amp. Hence the current is $\cdot540\ \overline{\diagdown}\,5{\cdot}1°$ amperes, agreeing very fairly with the $\cdot535\ \overline{\diagdown}\,4{\cdot}8°$ amperes obtained graphically on p. 151.

In the following table and curves these quantities have been calculated and plotted (Fig. 101) for a combination of a choking coil of ·7 henry and 30ω resistance, in parallel with a 10-mfd. condenser (which we studied in series on p. 134), as the frequency

TABLE XV.—COMBINATION OF CHOKING COIL AND CONDENSER IN PARALLEL.

$\bar{V} = 200$, $L =$ ·7 henry, $r = 30\omega$, $K = 10$ mfds. $= 10^{-5}$ farads.

Frequency $n \sim$.	$p = 2\pi n$.	Reactance of Coil $x_a = Lp$.	Impedance of Coil $I_a = \sqrt{r_a^2 + x_a^2}$ Ohms.	Conductance of Coil $g_a = \frac{r_a}{I_a^2}$ Mhos.	Susceptance of Coil $b_a = \frac{x_a}{I_a^2}$ Mhos.	Admittance of Coil $Y_a = \frac{1}{I_a}$ Mhos.	Current in Coil $\bar{C}_a = Y_a V$ Amperes.	Tan $\phi_a = \frac{b_a}{g_a}$.	Admittance or Susceptance of Condenser $= Y_b = b_b = -Kp$.	Current in Condenser $\bar{C}_b = Y_b V$.	Total Susceptance $b = b_a + b_b$.	Total Admittance $Y = \sqrt{g^2 + b^2}$.	Total Current $\bar{C} = YV$.	Tan $\phi = \frac{b}{g}$.	ϕ Degrees.
0	0	0	30	·0333	0	·0333	6·67	0	0	0	0	·0333	6·67	0	0
10	62·8	43·9	53·2	·0106	·0155	·0188	3·56	1·46	·000628	·1256	·0149	·0183	3·661	1·406	54·6
20	125·6	87·8	92·8	·00349	·0101	·0108	2·16	2·90	·001256	·2512	·00984	·0104	2·08	2·820	70·5
30	188·4	131·9	135·2	·00164	·00721	·00740	1·48	4·40	·001884	·3768	·00533	·00557	1·114	3·248	72·9
40	251·2	175·6	178·1	·000946	·00554	·00566	1·132	5·85	·002512	·5024	·00303	·00317	·634	3·203	72·7
50	314	219·5	221·6	·000612	·00447	·00452	·904	7·30	·00314	·6283	·00133	·00147	·294	2·175	65·3
60	376·8	263·8	265·5	·000426	·00374	·00377	·754	8·78	·003768	·7536	−·00003	·00043	·086	−·705	−3·52
70	439·6	307·7	309·2	·000315	·00322	·00324	·648	10·21	·004396	·8792	−·001176	·00122	·244	−3·736	−75
80	502·4	351·2	352·5	·000242	·00283	·00284	·568	11·69	·005024	1·005	−·00219	·00220	·440	−9·055	−83·7
90	565·2	395·1	396·2	·000192	·00252	·002525	·505	13·12	·005652	1·130	−·00313	·00314	·628	−16·30	−86·5
100	628	439	440	·000155	·00227	·002274	·455	14·63	·00628	1·256	·00401	·00401	·802	−25·87	−87·8
150	942	658·5	659·2	·000069	·001515	·001517	·3034	21·95	·00942	1·884	·00790	·00790	1·580	−114·4	−89·5
200	1256	878	878·5	·000039	·001138	·001139	·2278	29·20	·01256	2·512	·01142	·01142	2·284	−293·3	−89·8
300	1884	1319	1319·4	·000017	·000758	·000758	·1516	44·55	·01884	3·768	·01808	·01808	3·616	−1063	−89·95
∞	∞	∞	∞	0	0	0	0	-	∞	∞	∞	∞	∞	∞	−90

is varied. It will be seen that the current, from being large at first and in phase, gets smaller and lagging as the frequency rises; but that after 40 ~, while the current continues to decrease,

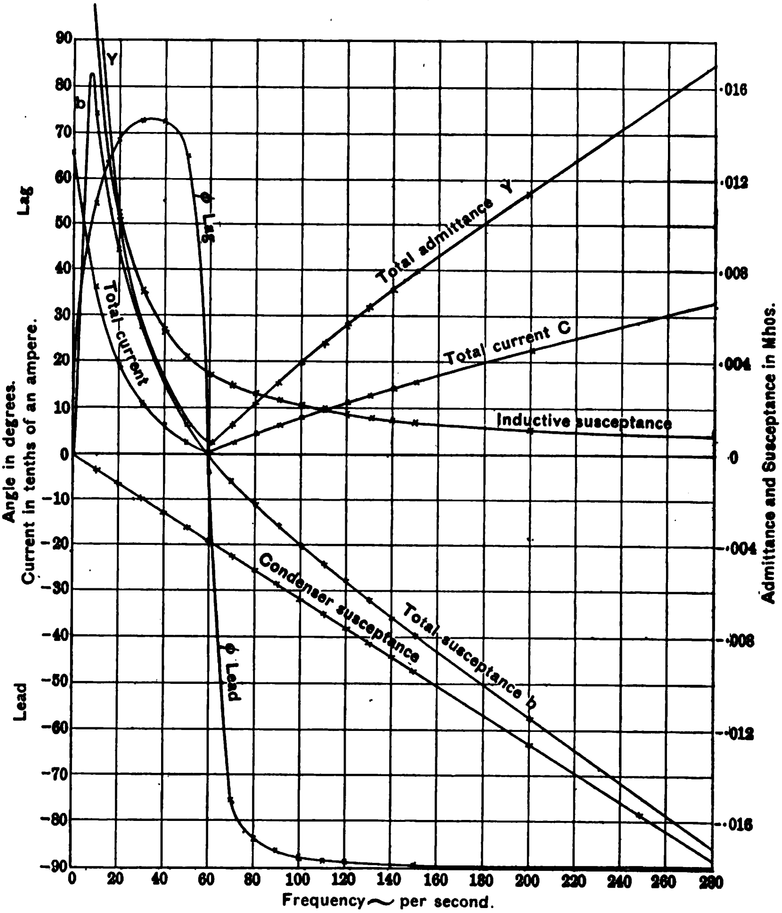

FIG. 101.—Variation of current, &c., for condenser and choking coil in parallel.

the lag also decreases. At a frequency of 60 periods, which is near to the 60·3 ~ for which $Lp = \frac{1}{Kp}$ (see p. 135), the current has dropped to ·086 amp. and the phase angle has actually begun to lead. After this the current and angle of lead increase together until at high frequency, where the choking coil practically stops all current, the main current is simply the condenser current and leads 90°.

Graphical Method of dealing with Circuits in Parallel.—On p. 150 it was shown how we could combine the currents in any number of circuits in parallel by means of a vector diagram. It was necessary, however, to first calculate the currents by the formula $\bar{C} = \frac{\bar{V}}{I}$ and $\tan \phi = \frac{x}{r}$. It is quite easy, by a simple linkwork, to find the resultant admittance, conductance, and susceptance, or impedance, resistance, and reactance, for any number of circuits in parallel without any calculation whatever; if the impedances of the separate branches are given either as $I \angle \phi$ or by their resistance r and reactance x. If we have the impedance given as $I \angle \phi$, we know that the admittance is $\frac{1}{I} \ulcorner \phi$, as the current lags behind the P.D. by the same angle as the P.D. leads in front of the current. We can easily add together impedances or admittances on the drawing board as vectors, but to convert from an impedance to an admittance, or *vice versa*, we must obtain the reciprocal. There are graphical ways of doing this, but the quickest way, if one needs to do much of this kind of work, is to employ the simple mechanism shown in Fig. 102.

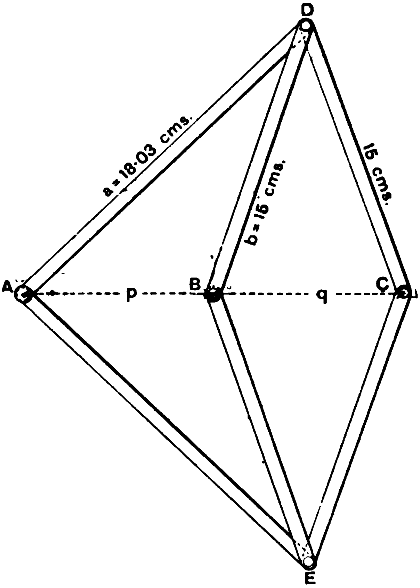

FIG. 102.—Peaucellier linkage.

It consists in the first place of four equal links pivoted together, which may conveniently be $b = 15$ cm. long between centres, and two longer ones, which should then be $a = 18{\cdot}03$ cm. between centres. The pivot A is provided with a needle point which forms an axis round which the whole arrangement can turn, B and C with indicating points, and D and E with smooth supports which can slide on the paper. Then it can easily be proved that however the mechanism is moved the product of the two distances $AB = p$ and $AC = q$ is constant and equal to $a^2 - b^2$. In our case $a^2 = 18{\cdot}03^2 = 325$, and $b^2 = 15^2 = 225$, so that $pq = a^2 - b^2 = 100$, or $p = \frac{100}{q}$. Hence when $p = 10$, $q = 10$, or the points p and q are together. If q is increased to 20 cm. B will move backwards towards A and will be $\frac{100}{20} = 5$ cm. &c. If we take our unit distance as 10 cm. or one decimetre we have simply $pq = 1$ or $p = \frac{1}{q}$. Hence if $q = I$, $p = \frac{1}{I} = Y$, and if $q = Y$, $p = \frac{1}{Y} = I$. As to phase, we have only to remember that angles above the horizontal are leading for impedances and lagging for admittances.

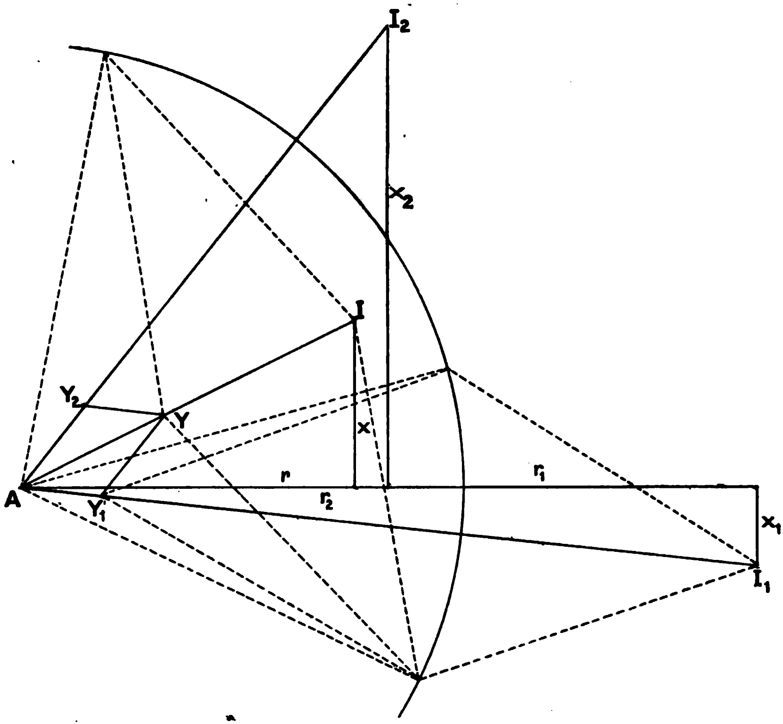

FIG. 103.—Graphical solution of impedances in parallel.

Fig. 103 shows the application of this method to the case of two circuits in parallel, one consisting of a 50-mfd. condenser in series with a resistance of 300ω, and the other of a choking coil of ·3 henry and 150ω on a 100 ~ circuit. We then have $r_1 = 300\omega$ and $x_1 = -31{\cdot}7\omega$, while $r_2 = 150\omega$ and $x_2 = 188{\cdot}5\omega$. Setting out these distances as shown in the diagram we have the two impedances I_1 and I_2. Putting the needle point of our linkwork at *A* and the point *C* at I_1 the point *B* immediately gives us Y_1 as $\cdot 00331\ \overline{\diagdown}\, 6^\circ\ 3'$ mho, while the two components are $g_1 = \cdot 0033$ mho and $b_1 = -\cdot 00035$ mho. Treating I_2 in a similar manner, $\mathrm{Y}_2 = \cdot 00415\ \angle 38^\circ\ 35'$, $g_2 = \cdot 00259$ mho, and $b_2 = \cdot 00325$ mho. Combining these together by completing the parallelogram, we have the resulting admittance $\mathrm{Y} = \cdot 00656\ \angle 26^\circ\ 15'$, while the resultant conductance $g = \cdot 00589$ mho and susceptance $b = \cdot 0029$ mho. Using our linkage again and putting *B* on Y, *A* gives us the resultant impedance I at once, which comes out as $152{\cdot}5\ \angle 26{\cdot}3^\circ$ ohms, while its components are $r = 136{\cdot}5\omega$, and $x = 67{\cdot}5\omega$. The combination is thus equivalent to a choking coil of resistance $136{\cdot}5\omega$ and inductance $\frac{x}{p} = \cdot 1072$ henry, at this frequency. By calculation on the lines given above we find $\mathrm{I} = 153\ \overline{\diagdown}\, 26^\circ$ ohms, $r = 135\omega$, and $x = 67{\cdot}1\omega$, which are within 1°/₀ of the values obtained by the linkage.

The linkage which we have just described was devised by Peaucellier for the drawing of inverse curves and as a parallel motion, and has many useful applications. It is most instructive to work out the variation of the current or admittance of a choking coil when the frequency alters, as in Fig. 82. We there saw that the impedance vector moved along a straight line and the current round a semicircle. Conversely, if the point *B* of the linkage moves along a semicircle, *A* must move along a straight line, and this is really the only known mathematical way of generating a straight line.

As a conclusion to this chapter the following table summarises the formulæ and mode of calculating circuits in series and parallel:

TABLE XVI.—COMBINATIONS OF RESISTANCE, INDUCTANCE, AND CAPACITY. (SUMMARY.)

—	Circuits in Series.	Circuits in Parallel.
Separate Circuits	Resistance r_a Reactance $x_a = L_a p - \frac{1}{K_a p}$ Impedance $I_a = \sqrt{r_a^2 + x_a^2}$ Lag :—tan $\phi_a = \frac{x_a}{r_a}$ $\bar{V}_a = I_a\bar{C} \angle \phi_a$ or $\bar{C}_a = \frac{\bar{V}}{I_a} \ulcorner \phi_a$	Conductance $g_a = \frac{r_a}{r_a^2 + x_a^2} = \frac{r_a}{I_a^2}$ Susceptance $b_a = \frac{x_a}{r_a^2 + x_a^2} = \frac{x_a}{I_a^2}$ Admittance $Y_a = \sqrt{g_a^2 + b_a^2} = \frac{1}{I_a}$ Lag :—tan $\phi_a = \frac{b_a}{g_a}$ $\bar{C}_a = Y_a\bar{V} \ulcorner \phi_a$ or $\bar{V}_a = \frac{\bar{C}}{Y_a} \angle \phi_a$
Combined Circuits	Total resistance $r = r^a + r_b + r_c$, &c. $= \Sigma r_a$ Total reactance $x = x_a + x_b + x_c$, &c. $= \Sigma x_a$ Total impedance $I = \sqrt{r^2 + x^2} = \sqrt{(\Sigma r_a)^2 + (\Sigma x_a)^2}$ Lag :— tan $\phi = \frac{x}{r} = \frac{\Sigma x_a}{\Sigma r_a}$ $\bar{V} = I\bar{C} \angle \phi$, $\bar{C} = \frac{\bar{V}}{I} \ulcorner \phi$	Total conductance $g = g_a + g_b + g_c$, &c. $= \Sigma g_a$ Total susceptance $b = b_a + b_b + b_c$, &c. $= \Sigma b_a$ Total admittance $Y = \sqrt{g^2 + b^2} = \sqrt{(\Sigma g_a)^2 + (\Sigma b_a)^2}$ Lag :— tan $\phi = \frac{b}{g} = \frac{\Sigma b_a}{\Sigma g_a}$ $\bar{C} = Y\bar{V} \ulcorner \phi$, $\bar{V} = \frac{\bar{C}}{Y} \angle \phi$
Result	Total conductance $g = \frac{r}{I^2} = \frac{r}{r^2 + x^2}$ Total susceptance $b = \frac{x}{I^2} = \frac{x}{r^2 + x^2}$ Total admittance $Y = \frac{1}{I} = \sqrt{g^2 + b^2}$	Total equivalent resistance $r = \frac{g}{Y^2} = \frac{g}{g^2 + b^2}$ Total reactance $x = \frac{b}{Y^2} = \frac{b}{g^2 + b^2}$ Total impedance $I = \frac{1}{Y} = \frac{1}{\sqrt{g^2 + b^2}} = \sqrt{r^2 + x^2}$

CHAPTER XII

POWER IN ALTERNATE-CURRENT CIRCUITS

One more fundamental matter concerning alternate-current theory remains to be dealt with, and that a most important one. How is the power supplied by an alternate-current circuit to be reckoned or measured? In the case of continuous currents we know that the power in watts is obtained by multiplying the P.D. in volts by the current in amperes; and we saw in Chapter IV that this was equivalent to obtaining the power required to move a body by multiplying the driving force in pounds by the velocity in feet per second.

Before making any calculations on the subject, however, we must first consider what we mean by the power in an alternate-current circuit. If, for example, we have a non-inductive load supplied from alternate-current mains, we know that the current and P.D. rise and fall together, and that at certain instants they are therefore both zero. It is hardly necessary to state that the flow of power at this instant is zero, while, when the P.D. and current are at their maximum values, the power supply is evidently considerable. The flow of power in an alternate-current supply is therefore irregular or pulsating, and we must define what we mean by 'the power' supplied.

For this purpose we must look at the matter from the utilitarian standpoint, as we have already done in the case of current and P.D. measurement. What we are concerned with as practical engineers is the amount of energy which has been supplied in any given time. Our electric light bills are made out on the number of units or kilowatt-hours supplied, as this is the actual amount of energy drawn from the mains, and derived from the coal or water power available. If we divide this number of units or kilowatt-hours by the total time in hours over which it has been supplied, we get the average power supply in kilowatts. It is clear therefore, from the practical point of view, that when the power supply is pulsating what we want is the ordinary average or mean value of the watts, as this, when multiplied by the time, will give us the total energy consumption. Thus in this case we measure the power by its mean value, and not by the square root of mean square as in the case of current or P.D. The square root of mean square in the latter cases was adopted because the mean power in watts is proportional to the mean *square* of the current in amperes or of the P.D. in volts, when the circuit is a resistance.

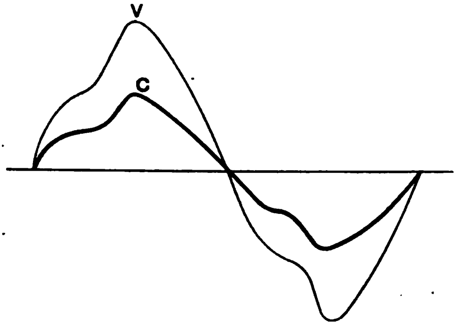

FIG. 104.—P.D. and current waves in phase.

We have therefore to find what the mean power supplied to a circuit is in watts when the P.D. and current are known. Now the power in watts *at any instant* is the product of the P.D. in volts, and the current in amperes at that instant, under all circumstances, but it does not by any means follow that the mean power in watts can be obtained by multiplying the ammeter reading or effective current in amperes by the voltmeter reading or effective P.D. in volts. We shall therefore proceed to find the relation between the power and the P.D. and current, but although it would be possible to do this for all cases at once, we shall find it convenient to start with two special cases: firstly, when the P.D. and current are in phase; and secondly, when they are in quadrature. Afterwards we can extend the reasoning to other cases.

CASE I. **P.D. and Current in Phase.**—Suppose we have an alternating P.D. as indicated in the fine curve V (Fig. 104), and a current indicated by the thick curve C rising and falling with it, and always exactly proportional to the voltage curve, whatever its form may be. In this case we may say that $V \propto C = aC$ where a is a constant.

This simply states that V is proportional to C, without necessarily implying that the circuit is a non-inductive resistance.

But if $V = aC$, $\bar{V} = a\bar{C}$, since in taking the mean square of the voltage curve each value is multiplied by a^2, and hence the $\sqrt{\text{mean square}}$ is multiplied by a.

To find the power w in watts at any instant we have $w = VC = aC \times C = aC^2$, or the watts are proportional to the square of the current, as we know.

Hence mean or average of $w = \bar{w} = \text{mean } aC^2 = a \times \text{mean } C^2$.

But mean $C^2 = \bar{C}^2$, since $\bar{C}$, the effective value of C, $= \sqrt{\text{mean } C^2}$.

Hence $\bar{w} = a \times \text{mean } C^2 = a\bar{C}^2$, or the mean watts are proportional to the square of the effective amperes, and this may be written $\bar{w} = a\bar{C}^2 = a\bar{C} \times \bar{C} = \bar{V}\bar{C}$, since $\bar{V} = a\bar{C}$.

We thus come to the following important and remarkable result:

When an alternating P.D. and an alternating current are similar to one another and in phase, the mean power supplied to the circuit in watts is equal to the product of the effective P.D. in volts and the effective current in amperes, whatever the wave form may be. Or, in other words, the mean watts are obtained by multiplying the ammeter and voltmeter readings, just as with continuous currents.

This proof is entirely independent of the wave form of the supply or of the nature of the load, but as a matter of fact the P.D. and current are rarely, if ever, exactly similar in form, except when the load is a non-inductive resistance, such as is given by glow-lamps. Wherever alternate-current mains are supplying glow-lamps, however, we may find the power by multiplying the volts and amperes indicated on the switchboard instruments just as if the supply was continuous.

Case II. **P.D. and Current in Quadrature.**—Turning now to our second case, suppose that we have an alternating P.D. and an alternating current both varying in a simple harmonic manner, but in quadrature, the current lagging behind or leading in front of the P.D. by 90°. What then is the average power supplied?

First take one or two mechanical illustrations. Since P.D. or electrical force corresponds to mechanical force, current corresponds to velocity, and electrical power, the product of P.D. and current, to mechanical power, which is the product of force and velocity, it is clear that if we find the average power supplied to an oscillating body when the force and velocity are in quadrature, we shall thereby find the power in an alternating-current circuit when the P.D. and current are in quadrature.

Now, when do we find force and velocity in quadrature? Perhaps the simplest case is that of the ordinary pendulum as in Fig. 49 or Fig. 105. If such a pendulum is set swinging, the greatest force towards the central position is when the pendulum is at its highest point, or at the end of the stroke. The greatest velocity in the same direction does not occur till the bob is at its lowest point, a quarter of a period later. We know that for small swings the force and velocity vary according to a simple harmonic law, and hence we have an example of a simple harmonic force, and a simple harmonic velocity lagging 90° behind it, as we saw on p. 119.

How much is the mean power in this case? Evidently very small, as the pendulum will go on vibrating for a very long time without any fresh impulse. But if the pendulum were suspended in a vacuum by an extremely fine fibre in which there was no friction, we know that it would go on vibrating for ever. If there is no friction, there is no dissipation of energy, and as there is no *resultant* storage of energy, since the bob is not permanently raised, there is no need for a continued supply of power. To be sure, the force of the earth's attraction does work on the bob as it moves from *A* to *C*, as the force is in the same direction as the motion, but as soon as the bob has passed *C* the force is reversed and the bob moves on in opposition to the force. Hence in passing from *C* to *B* the bob does work instead of receiving it—it is a 'motor' instead of a 'dynamo.' On going back from *B* to *C* the force and velocity are in the same direction and the bob receives energy again, but from *C* to *A* the bob does work against gravity once more. In electrical language, therefore, the pendulum is a dynamo or generator twice and a motor twice in each period, and the one compensates for the other, so that the mean power received or given out is zero.

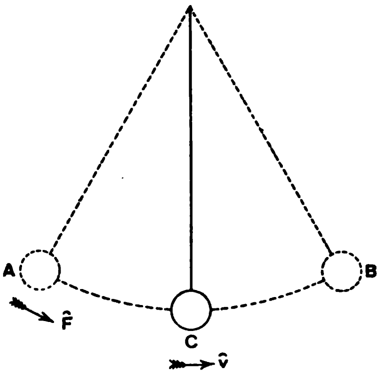

FIG. 105.—Pendulum.

It will, of course, be said that in practice the pendulum oscillations will die away, thus giving up the energy. But this means friction, and any friction means that there must be a component of the velocity in phase with the force, so that we have not got perfect quadrature. Perfect quadrature can only exist when there is no friction, and hence no frittering away of energy; in this case, however great the force or the motion may be, there is no average power.

We may therefore accept it as true that, when a simple harmonic velocity is produced by a simple harmonic force but lags 90° behind it in the case of a moving mass or pendulum, the average power absorbed or supplied is nil. But this simply means in electrical language that when we apply a simple harmonic P.D. to an electrical mass or inductance, obtaining, as we know, a current lagging 90° if there is no resistance or electrical friction, there can be no average expenditure of power. While the current is increasing in the first part of the period, it is producing a magnetic flux, and storing energy in the medium around the coil, but when the current falls again, and the lines of force get fewer, they cut the coil and give back their energy to the circuit, as the bob gave back its energy of motion on the up-stroke. We therefore have the condition that, although the effective P.D. as shown by a voltmeter and the effective current as shown by an ammeter may each of them be very large, the circuit does not absorb any power if they are exactly in quadrature. If there is the smallest amount of resistance, there will be an absorption of power in forcing the current through it, but the current will then no longer lag exactly 90° behind the P.D. Inductance by itself wastes no appreciable energy.*

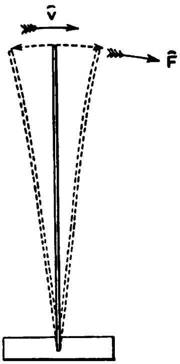

FIG. 106.—Spring.

To see what occurs when the current or motion leads we have only to consider a spring. Take a springy piece of steel, as in Fig. 106, and bend the end of it backwards and forwards with a simple harmonic motion. In this case, as before, the force is greatest at the end of the stroke, but it is now away from the centre instead of towards it as in the pendulum. The maximum velocity is at the centre of the stroke or is 90° earlier than the maximum force in the same direction, which does not exist until the end of the stroke. The velocity is therefore in quadrature with the force as before, but leads, as we saw in Chapter IX.

It is obvious again in this case that if the spring is perfect, and there is absolutely no friction, the mean power supply is nil. Energy is required to bend the spring to one side, but this is given back on the recoil, so that twice in each period the spring receives energy and twice returns it.

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* Except at very high frequencies when it produces radiation in the surrounding medium (see Chapter XVII).

Now this corresponds to a perfect condenser in which there is neither leakage nor absorption. In this case we have a leading current in quadrature, and it is evident that the condenser is charged and discharged, but that there is no dissipation of energy.

We are therefore prepared to expect that in all cases where the P.D. and current are in quadrature the mean power is zero, but a more general proof is still required. Such a proof can be given by reference to the diagram (Fig. 107), which illustrates a simple harmonic P.D. V and a simple harmonic current C lagging by 90°. These curves would be produced by the rotating vectors V and C on the left-hand side, which are at right angles to one another.

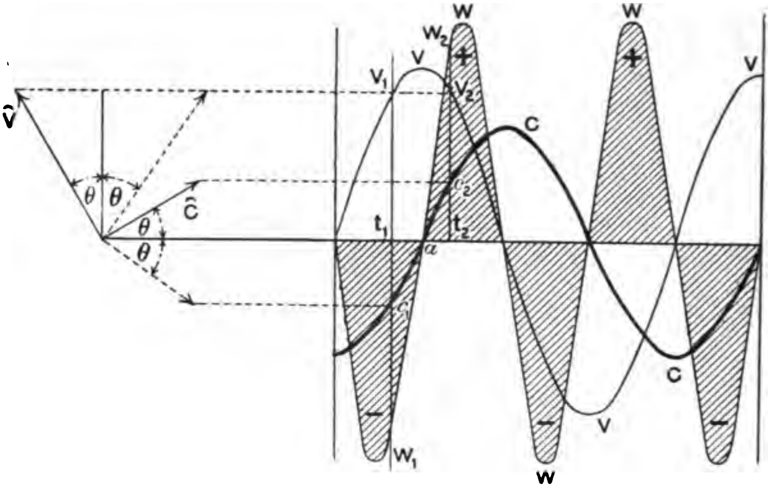

FIG. 107.—P.D., current, and power curves.

At the instant represented by t_1 the current vector makes an angle θ downwards from the horizontal, and the voltage vector must similarly make an angle θ with the vertical. The values t_1C_1 and t_1V_1 are the instantaneous values of the current and P.D. at that instant, and their product t_1w_1 the value of the watts at the same instant, which is here negative, as the current is in the opposite direction to the P.D. If we now take a time t_2 such that the current vector makes the same angle with the horizontal, but upwards, it is evident that the instantaneous value of the current t_2C_2 will be the same as before, but positive, while the P.D. t_2V_2 will be the same as t_1V_1. The new value of the power in watts t_2w_2 will be the same as before in amount, but positive, or it is equal and opposite to t_1w_1. By the same reasoning we see that at any instant of time after a (when the current vector is horizontal) the power is equal and opposite to its value at an equal interval of time before a, and that therefore the positive and negative ordinates of the power curve are equal for equal times. But this evidently implies that the mean value is zero, for if the ordinates of the power curve at equal intervals are added up in order to take the mean, the positive and negative values will cancel one another. This reasoning evidently holds whether the current is lagging or leading, and we may therefore say:

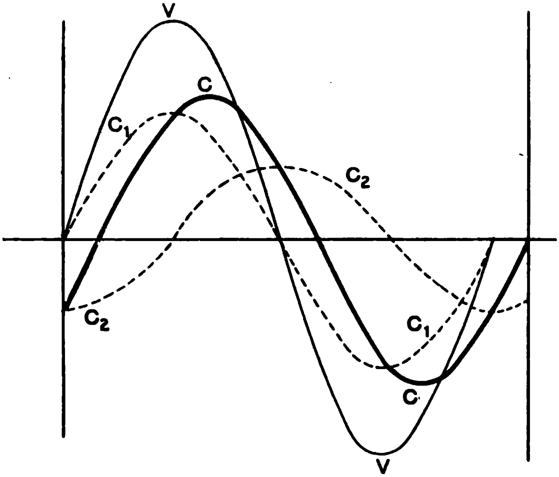

FIG. 108.—P.D. and lagging current curves with energy and wattless current components.

Whenever two simple harmonically varying quantities are in quadrature, their mean product is zero.

Or for alternate currents, *whenever the P.D. and current in an alternate-current circuit are simple harmonic and in quadrature, the mean power is zero.*

P.D. and Current in any Phase Relation.—It has now been shown that when the P.D. and current are in phase $\mathfrak{w} = \bar{V}\bar{C}$, and that when they are in quadrature $\mathfrak{w} = 0$. It is now perfectly easy to see what the value of the mean power must be, when a circuit is supplied from a simple harmonic P.D. with a simple harmonic current, in any phase relation. For suppose, as in Fig. 108, we have a P.D. and current represented by the curves V and C with an angle of lag much less than 90°. This may be represented, as we know, by Fig. 109, in which the vector $\bar{V}$ represents the effective value of the P.D., and the vector $\bar{C}$ the corresponding effective current. But we have already learnt that simple harmonic quantities represented either by their maximum or effective values as vectors may be compounded or split up, just like forces, and we accordingly split up the current into two parts $\bar{C}_1$ and $\bar{C}_2$, the first in phase with V and the second in quadrature with it, as was done when parallel circuits were being considered. We may then consider that we have a P.D. of effective value $\bar{V}$ supplying two circuits, one with a current of effective value $\bar{C}_1$ in phase with V, and the other with a current $\bar{C}_2$ in quadrature with it. But by the results just obtained the mean power supplied to the first circuit would be $w = \bar{V}\bar{C}_1$, and to the second zero. Hence the total power supplied will be $w = \bar{V}\bar{C}_1 = \bar{V}\bar{C} \cos \phi$ where ϕ is the angle of lag.

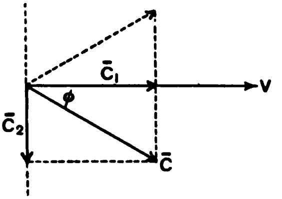

FIG. 109.—Energy and wattless currents.

This result is of the greatest possible importance, and the proof should be thoroughly assimilated. It will be seen at once that the formula is in agreement with the results we obtained in special cases; for if the P.D. and current are in phase, $\phi = 0$ and $\cos \phi = 1$, whence $w = \bar{V}\bar{C}$, as we saw. If, on the other hand, $\bar{V}$ and $\bar{C}$ are in quadrature, $\phi = \pm \frac{\pi}{2}$ and $\cos \phi = 0$, from which $w = 0$, as above.

In the case of an ordinary impedance $I = \sqrt{r^2 + x^2}$ where $x = Lp - \frac{1}{Kp}$, we have evidently $\cos \phi = \frac{r}{I}$ (from Fig. 99) and $\bar{V} = I\bar{C}$. Hence $w = \bar{V}\bar{C} \cos \phi = I\bar{C} \times \bar{C} \times \frac{r}{I} = \bar{C}^2 r$, since I cancels, which is exactly the result for the current $\bar{C}$ flowing through the resistance only. Thus inductance or capacity may alter the current, but neither of them absorb any power, as above indicated.

Apparent Watts and Power Factor.—Since the product of the effective P.D. and current, or $\bar{V}\bar{C}$, would give the power in watts in the circuit if the supply were direct current or if i were alternate current in phase, it is customary to call it the apparent watts and to denote it by W. Hence we have—

$$\text{True power } \bar{w} = \bar{V}\bar{C}\cos\phi = W\cos\phi \quad . \quad . \quad (36)$$

or

$$\cos\phi = \frac{w}{W} = \frac{\text{true watts}}{\text{apparent watts}} \quad . \quad . \quad (37)$$

Thus the true power in an alternate-current circuit is obtained by multiplying the product of the P.D. in volts and the current in amperes (or the apparent watts), by a factor which is the cosine of the angle of phase difference. But if the current and P.D. wave forms are not simple harmonic, or if they do not resemble one another, there is no definite meaning to the term '*angle of lag*.' It will be readily seen, however, that we may still obtain the true power in the circuit from the apparent power, by multiplying by some factor k which cannot be greater than unity.* We thus have $w = k\bar{V}\bar{C} = kW$, where k is this factor and is termed the *power factor* of the circuit. When the wave forms are sinusoidal $k = \cos\phi$, but when they are not, k is called the *power factor* of the circuit and is simply that factor which, when multiplied by the product of the effective volts and amperes, gives the true mean power in watts. Hence we have $\bar{w} = k\bar{V}\bar{C} = kW$, whatever the wave form may be,

or $k = \frac{\bar{w}}{W}$, and whenever the wave form is sinusoidal, $k = \cos\phi$.

For this reason the power factor of a circuit is often colloquially spoken of as its '$\cos\phi$,' and as a matter of fact, even when the wave forms are irregular, we can get the power factor experimentally by measuring the true watts $\bar{w}$ and the effective P.D. and current $\bar{V}$ and $\bar{C}$, and thence $k = \frac{\bar{w}}{\bar{V}\bar{C}}$, which is always unity or less. We can therefore always find an angle from a table of cosines such that its cosine is equal to k, and this angle has been termed the *angle of power lag*, which is the true angle of lag when the wave forms are sinusoidal.

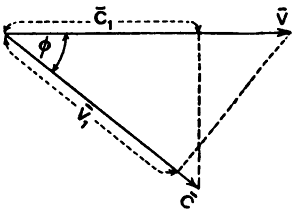

FIG. 110.—Scalar product of P.D. and current.

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* Cos ϕ cannot of course be greater than unity, but this does not necessarily prove that k cannot be greater than 1, when the wave forms are not simple harmonic. We have seen, however, that $\bar{w} = \bar{V}\bar{C}$, or that $k = 1$ when the P.D. and current are in phase, whatever the wave form; and it is tolerably evident that any displacement of phase must reduce the power.

It should be noticed that the power and power factor are the same for a given phase difference between the P.D. and current, whether the latter is lagging or leading. This can be seen at once by drawing another current vector, shown dotted in Fig. 109, leading by the same angle ϕ.

Comparison between Alternate-Current and Mechanical Power.—Reverting to sinusoidal currents, it is interesting to notice a simple mechanical analogy to the result just obtained, viz. $\bar{w} = \bar{V}\bar{C} \cos \phi$. This means, as we saw in Fig. 109, that the mean power $\bar{w}$ is the product of $\bar{V}$ and of the energy component of $\bar{C} = \bar{C} \cos \phi$, obtained by dropping a perpendicular from $\bar{C}$ upon $\bar{V}$. But it will also be noticed that this result is the same as dropping a perpendicular from the end of $\bar{V}$ upon $\bar{C}$, and multiplying $\bar{C}$ by the component of $\bar{V}$ in the direction of $\bar{C} = \bar{V} \cos \phi$. Hence, as in Fig. 110, if the voltage and current vectors are represented by $\bar{V}$ and $\bar{C}$, the power $\bar{w} = \bar{V}\bar{C}_1 = \bar{C}\bar{V}_1 = \bar{V}\bar{C} \cos \phi$. The product of one vector and the component of another in its direction is known as the *scalar product* of two vectors, and hence we may say that with simple harmonic P.D.'s and currents, the power supplied to a circuit is the scalar product of the P.D. and current of that circuit.

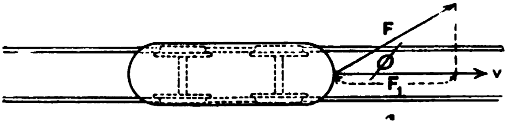

FIG. 111.—Force and motion in different directions.

But now consider the case of a horse drawing a tramcar with uniform velocity. We saw in Chapter IV that the power employed in moving the car was the product of the force and velocity, or $P = Fv$. But this is only true when the force is in the direction of the motion of the car. If, for example, the horse were attempting to pull the car at right angles to the rails, he would be doing no work on the car, even if it were moving, as his force would neither help nor retard the motion of the car along the rails. Hence it is not the product of the force and velocity which is the power if the pull is not in the direction of the motion, but the product of the velocity and the component of the force in the direction of the velocity, or of v and F_1 in Fig. 111. But if ϕ is the angle which F makes with v, $F_1 = F \cos \phi$, and the power $P = Fv \cos \phi$, which is exactly analogous to $\bar{w} = \bar{V}\bar{C} \cos \phi$ for the alternate-current circuit. In other words, the mean power in an alternating-current circuit is the product of the effective electric force in volts and effective electric velocity in amperes, and the cosine of the angle of phase difference between them; while the power exerted in moving a mass is the product of the mechanical force and the mechanical velocity, and the cosine of the angle of inclination between them.

Measurement of Alternate-Current Power.—This is a subject which must be dealt with more fully elsewhere, but two simple examples may be taken, as they will help us to thoroughly realise the principles. We will deal with the wattmeter and three-voltmeter methods of measuring the power supplied to an alternate - current circuit. The wattmeter is an instrument which automatically performs the operation of multiplying together the P.D. in volts and the current in amperes at every instant and indicates the mean value of this product, or the true average power. There are various ways in which this result may be attained; but one of the most obvious is to make use of the dynamometer principle. If two coils are placed near to one another and steady currents C_1 and C_2 are passed through both of them, they will attract or repel one another, or one will tend to turn inside the other, with a force or torque proportional to the product of C_1 and C_2. If the currents vary rapidly in a periodic manner so that the movement of the coils has not time to follow the variations, the average force or torque only will be measured, and will be proportional to the average product of C_1 and C_2. Now if we pass the main alternating current through one of the coils, and send a current through the other coil, which is rigidly proportional to the P.D., as is

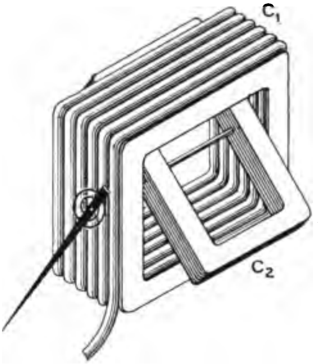

FIG. 112.—Dynamometer wattmeter.

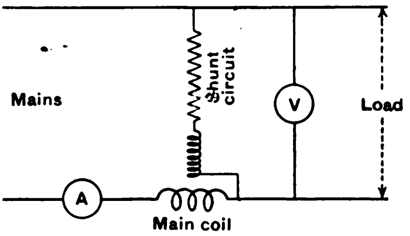

FIG. 113.—Connections of wattmeter.

the case with a non-inductive resistance, then the mean product of C_1 and C_2 is proportional to the mean product of C_1 and V, or to the average power. The dynamometer wattmeter therefore usually consists of a fixed main coil, which carries the current of the supply, and a moving coil of fine wire, which is connected in series with a high non-inductive resistance across the mains. The moving coil is generally suspended or pivoted so as to turn in the field of the main coil, as in Fig. 112, and the force is indicated by the movement of a pointer against a spring control. It will be seen that the wattmeter is thus equivalent to a combined ammeter and voltmeter, the two circuits being connected precisely as the separate ammeters and voltmeters in Fig. 113. It will also be evident that for the indications of a wattmeter to be accurate there must be no appreciable lag or lead of the current in the moving coil behind the P.D.; but as the coil always has some inductance this will not be the case, unless the non-inductive resistance is very high in comparison with the reactance of the coil. The consideration of the errors thus produced will, however, be deferred.

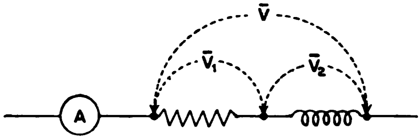

FIG. 114.—Three-voltmeter method.

If a wattmeter is joined to a circuit together with an ammeter and voltmeter, as in Fig. 113, it should be found that when the load is non-inductive, as, for instance, a bank of glow-lamps, the ammeter and voltmeter readings when multiplied together equal the wattmeter reading; but on condensers, choking coils, or motors the wattmeter reading is decidedly less than the product of volts and amperes and the power factor

$$k \text{ or } \cos\phi = \frac{\bar{w}}{\bar{V}\bar{C}}.$$

Three-voltmeter Method.—Although greatly inferior in accuracy to the wattmeter, this method, which is due to Professors Ayrton and Sumpner, is of considerable interest. The load in which the power consumption is to be measured is connected in series with a non-inductive resistance and ammeter to the mains, as in Fig. 114, and a single voltmeter or three voltmeters are used to obtain the three effective voltages $\bar{V}_1$ across the resistance, $\bar{V}_2$ across the load, and $\bar{V}$ across the resistance and load together. With direct currents, of course, V would always be equal to $V + V_2$, but with alternating currents the effective value $\bar{V}$ will be found to be less than $\bar{V}_1 + \bar{V}_2$ whenever the load is reactive. In fact, as the resistance and load are in series, we know that $\bar{V}$ is the vector sum of $\bar{V}_1$ and $\bar{V}_2$, which will only be equal to $\bar{V}_1 + \bar{V}_2$ when they are in the same phase. In all other cases $\bar{V}_1$, $\bar{V}_2$, and $\bar{V}$ must form the three sides of a triangle. Hence we can form the vector diagram of the voltages (Fig. 115) by laying down one of them, say $\bar{V}_1$, to scale, and striking off arcs of radii $\bar{V}_2$ and $\bar{V}$ from its two ends, thus forming a triangle with the three sides $\bar{V}_1$, $\bar{V}_2$, and $\bar{V}$. But $\bar{V}_1$ was the P.D. across the non-inductive resistance, which must be in phase with $\bar{C}$. We can then set off a vector representing $\bar{C}$ to scale from the end of $\bar{V}_1$ and in line with it, and we have $\bar{V}_2$ the P.D., and $\bar{C}$ the current vectors, for the load, set off with their proper phase relations; and the angle between them must be the angle of lag ϕ. We can scale off ϕ with a protractor, and we then have $w = \bar{V}_2\bar{C} \cos \phi$; or we can drop a perpendicular from the end of $\bar{C}$ upon $\bar{V}_2$, and the product of $\bar{V}_2$ with the component of $\bar{C}$ in its direction, or $\bar{V}_2\bar{C}_1$, will be the true power.

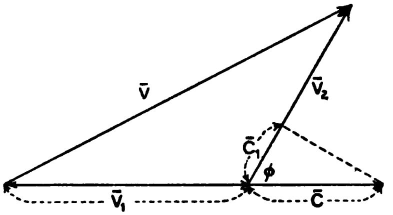

FIG. 115.—Vector diagram of 3-voltmeter method.

This solution is, of course, based on the assumption that the voltages and currents are simple harmonic, but we can obtain it for all wave forms. For, however the voltages may vary, we know that at each instant $V = V_1 + V_2$. But our voltmeters read the square root of mean square of the P.D. in each case. Squaring both sides of the equation, we therefore have $V^2 = V_1^2 + V_2^2 + 2V_1V_2$.

But if r is the value of the non-inductive resistance, $V_1 = Cr$ at every instant. Hence $V^2 = V_1^2 + V_2^2 + 2rV_2C = V_1^2 + V_2^2 + 2rw$. Taking the average value of both sides, and remembering that mean $V^2 = \bar{V}^2$, &c., and that mean $w = \bar{w}$, we have $\bar{V}^2 = \bar{V}_1^2 + \bar{V}_2^2 + 2r\bar{w}$, from which

$$\bar{w} = \frac{\bar{V}^2 - \bar{V}_1^2 - \bar{V}_2^2}{2r} = \frac{\bar{C}}{2\bar{V}_1}(\bar{V}^2 - \bar{V}_1^2 - \bar{V}_2^2)$$

It will be seen that when the P.D.'s and current are simple harmonic, this result is the same as we obtained from the vector diagram. For by ordinary trigonometry we know that

$\bar{V}^2 = \bar{V}_1^2 + \bar{V}_2^2 + 2\bar{V}_1\bar{V}_2 \cos\phi$, and hence $\bar{V}^2 - \bar{V}_1^2 - \bar{V}_2^2 = 2\bar{V}_1\bar{V}_2 \cos\phi$, which when multiplied by $\frac{\bar{C}}{2\bar{V}_1} = \bar{V}_2\bar{C}\cos\phi$, as before.

Positive and Negative Power: Generator and Motor.—We have seen that the power given by an alternate-current generator to a circuit is $\bar{w} = \bar{V}\bar{C}\cos\phi$, or $\bar{V}\bar{C}_1$, where ϕ is the angle of lag or lead and $\bar{C}_1$ the component of $\bar{C}$ in the direction of $\bar{V}$, and that when ϕ is 90°, or $\bar{V}$ and $\bar{C}$ are perpendicular, the average power is therefore zero. Now comes the question as to what would happen if the lag or lead were greater than 90°. It is evident from Fig. 116 that the component of $\bar{C}$ in the direction of $\bar{V}$ is in the opposite direction to $\bar{V}$ or is negative, or $\cos\phi$ is negative, so that the average power is also negative. This means that instead of the generator giving power to the circuit it receives power from it. This will be further realised by plotting the curves of V and C and the product w, as in Fig. 117, from which it is evident that the negative loops are larger than the positive ones, and that there is therefore a net balance of power returned from the circuit. What does this mean?

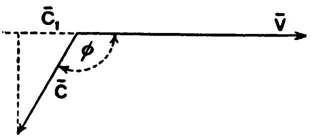

FIG. 116.—Current lagging more than 90°.

One of the clearest ways of realising this is to consider the case of a direct-current generator running in parallel with another machine which develops a constant E.M.F. or with an accumulator battery. If the first generator is fully excited and the speed is sufficiently high, its E.M.F. E_1 will exceed the E.M.F. of the other machine or battery E_2 (Fig. 118). There will therefore be a resultant forward E.M.F. $= E_1 - E_2$ and a current $C = \frac{E_1 - E_2}{r}$ in the direction of E_1. The power supplied by the generator will be $E_1 C$. Now suppose that the engine driving the generator slows down. The forward E.M.F. E_1 will get smaller while E_2 remains constant; hence $E_1 - E_2$ and C and the power supplied get rapidly less. If the speed drops until $E_1 = E_2$, there is no current and no supply of power, and if the speed drops further E_1 is less than E_2; hence $E_1 - E_2$ and C are negative, or the current reverses direction. But this means that the second machine or battery is discharging back or supplying power to the dynamo, which then runs as a motor. The more the dynamo is mechanically loaded as a motor, the slower it runs, and the larger is the current and return of power from the other machine. In other words, the change from generator to motor or from positive to negative power takes place by the shortening and reversing of the current vector, as in Fig. 118.

Now take the alternating-current case of a generator of E.M.F. $\bar{E}_1$ running in parallel with a similar machine of E.M.F. $\bar{E}_2$, the exciting currents of both machines being kept constant. It is fairly obvious that alternating-current machines will not run together unless their speed is the same, as if there were any difference one machine would sometimes have an E.M.F. in the opposite direction to the other, and they would short-circuit each other. Hence, if the second machine is kept at a constant speed, the generator must also always run at the same speed, just as two toothed wheels must always run together. Suppose that the E.M.F. of the generator $\bar{E}_1$ exceeds that of the second machine $\bar{E}_2$, then there will be a resultant E.M.F. equal to the *vector* difference $\bar{e}$ between $\bar{E}_1$ and $\bar{E}_2$, which will produce a current $\bar{C}$ equal to this vector difference divided by I, the impedance of the two machines in series. If $\bar{E}_1$ is larger than $\bar{E}_2$ and in phase with it, and the machines had no inductance, $\bar{C} = \frac{\bar{e}}{r}$, and is in phase with each of them exactly as in the direct-current case, and there will be a supply of power from the generator $\bar{w} = \bar{E}_1\bar{C}$, since $\bar{E}_1$ and $\bar{C}$ are in phase. But now suppose, as before, that the engine driving the generator

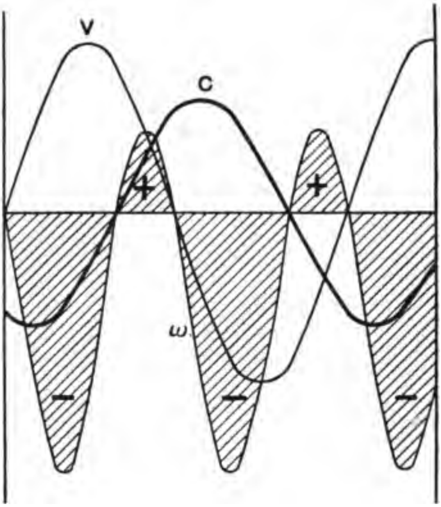

FIG. 117.—Current lagging more than 90°.

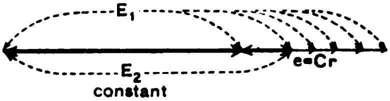

FIG. 118.—D.C. machines in parallel. Change from generator to motor.

drives less strongly. The speed will not fall in this case, as it must remain that of the other machine, and since the exciting current is kept constant $\bar{E}_1$ should not alter. But if $\bar{E}_1$ and $\bar{E}_2$ both remain the same, it would seem at first sight that the current must remain the same, and hence the power supplied by the generator, and yet the engine cannot supply it. What is the explanation of this paradox? The answer is that, as the engine develops less driving torque, the armature of the generator will tend to slow down. It cannot run at a lower speed, however, but will lag behind and the E.M.F. $\bar{E}_1$ will lag behind $\bar{E}_2$, as in Fig. 119. This means that the vector difference $\bar{e}$, and consequently the current $\bar{C}$, lags, and the power supplied by the generator $\mathfrak{w} = \bar{E}_1\bar{C}\cos\phi$ is less. In other words, the power has become less, not by $\bar{C}$ getting smaller, as in the direct-current case (Fig. 118), but by its altering in phase or turning to one side. As the engine power falls more and more, the lag of $\bar{E}_1$ and of $\bar{C}$ gets greater and the current rapidly increases in magnitude. If, however, the excitation of the generator is reduced so as to keep the current $\bar{C}$ at about the same value, it will be found that the current lags more and more until, when $\bar{C}$ is perpendicular to $\bar{E}_1$, the generator supplies no power. Going beyond this and actually cutting steam entirely off the engine, the current will lag still further behind quadrature; the power then becomes greater and greater in the negative direction, the second machine supplying power back to the generator and driving it as a motor. The change from generator to motor has, however, taken place by the current vector having passed beyond the quadrature position, and not by its first becoming zero, as with direct currents.

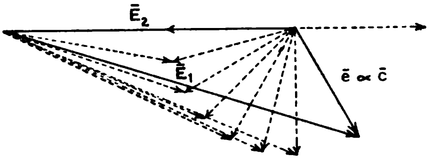

FIG. 119.—A.C. machines in parallel. Change from generator to motor.

It will be noticed that the current vectors in Fig. 119 represent the current supplied from machine 1 to machine 2, and therefore when reversed show the current supplied from machine 2 to machine 1. But this means that as soon as the current vector passes the quadrature position and power is supplied from machine 2 to machine 1, the current, instead of being considered as passing from machine 1 to machine 2 and lagging more than 90°, is more rationally taken as passing from machine 2 to machine 1 and *leading* by some angle less than

90°. A synchronous motor can therefore act as a condenser, as stated on p. 155.

The difference between the direct-current and alternate-current example of change from generator to motor can be well illustrated by the difference between pulling the truck on rails, as in Fig. 111, by a locomotive or by a horse. In the case of a locomotive it runs on the same rails, so that its driving force must always be in the direction of the track. When the truck is running the locomotive behaves as a generator or as a motor, depending upon whether it pulls the truck forward or pushes back against its motion, and the change from generator to motor takes place as the pull is reduced to zero before reversing, as with direct currents. The horse, however, is not confined to the track, and if he pulls with a constant force in different directions, like the different current vectors in Fig. 119, he will be a 'generator' or 'motor' depending upon whether the component of his pull along the track is forwards or backwards, and he changes over from one to the other as the rope passes the perpendicular or 'quadrature' position, as in the alternate-current case.

CHAPTER XIII

SYMBOLIC METHODS OF VECTOR AND ALTERNATE-CURRENT CALCULATION

We have now arrived at the point where we can find out what occurs in any system of alternate-current circuits by drawing vector diagrams, and by calculating the impedances or admittances of each portion of the system. This is really sufficient for all practical purposes, but the calculations are made more rapid and convenient, and at the same time more suitable for obtaining general formulæ, by adopting some means of expressing the vectors in algebraic form. There are two ways of doing this, both of which are convenient.

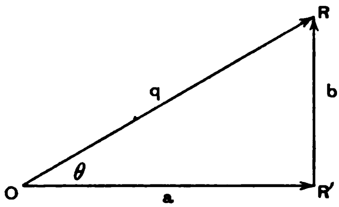

Fig. 120.—Components of a vector.

How can we write down a vector quantity in symbols? In Fig. 120 we have a vector *OR* having a length $= q$, and inclined at an angle θ with the horizontal. It is evident that we can write down this vector, as we have already often done, as $q \angle \theta$, and this form is very convenient. But it can also be expressed in another way, by dropping a perpendicular *RR'* upon the horizontal and saying that the vector is made up of two components, a horizontal component *OR'* or *a*, and a vertical component *R'R* or *b*, just like the resistance and reactance components of an impedance. For instance, the impedance of a coil having a resistance of 10 ohms and a reactance of 5 ohms could simply be stated as a vector having a horizontal component of 10 and a vertical component of 5, and this would be the most convenient mode of expression whenever we have circuits in series where the resistances and reactances have to be added separately.

If, therefore, we could have symbols to represent horizontal and vertical directions, we could write the impedance of the above coil as $10H + 5V$, and there is a method of vector calculation which uses symbols of this kind for the unit vectors along the horizontal and vertical directions. This could also be very conveniently used here, but as in ordinary alternate-current problems we are only concerned with vector diagrams on a flat surface and not in various directions in space, we can save trouble in writing by taking the horizontal direction as standard and simply writing the impedance as $10 + 5j$ where j is a symbol which is always used to express a direction at right angles, as the vertical is to the horizontal.

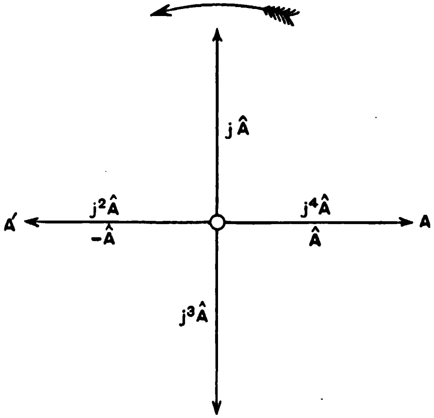

FIG. 121.—Turning of a vector.

Meaning of j.—Now what is meant by a symbol which expresses a direction at right angles? This brings us to a point at which a little imagination is necessary. How can the twisting round of a line or vector quantity be expressed? In Fig. 121 we have a vector OA, which we will denote by the symbol Â. Now if we draw a vector OA' exactly equal to the first but turned in the opposite direction, we may call it — Â. The reason for using the negative sign is a perfectly simple and practical one, for if Â were a force, then an equal force in the opposite direction would destroy or neutralise it just as +10 and —10 neutralise one another. Hence, turning a vector Â end for end or turning it through two right angles is equivalent to making it its own negative, or — Â, or to multiplying it by — 1. We may therefore say that the multiplying of a vector quantity by —1 is equivalent to turning it through two right angles.

Hence we have come to the important conclusion that we can express the turning of a vector quantity through two right angles by multiplying it by the quantity —1. We may then ask the question, Is there anything by which we can multiply a vector quantity so that the result will be to turn the vector through *one* right angle? Now it is certainly difficult to imagine what such a quantity could be, if it exists. In higher mathematics, however, one of the chief ways of finding out new quantities is to suppose they exist first and to give them a symbol, and then to find out what they must be like afterwards. We will therefore proceed by supposing there is a certain quantity to which we will give the symbol j, which when multiplied by any vector turns it through a right angle. In other words, if we have a vector $\hat{A}$, then $j\hat{A}$ will represent the same vector twisted round at right angles, or a vector of the same length as $\hat{A}$ but at right angles to it. Now, when we speak of its being at right angles, of course it might be either upwards or downwards. So we must agree first of all which to take. Prof. Steinmetz, to whom the development of the symbolic method is principally due, has taken it as lagging, and many writers have followed him, but it seems most natural, and indeed it appears to work out most simply in practice, to take the turning as upwards or leading, as is done in trigonometrical books. Hence, at the risk of criticism and of confusion with some standard treatises, we will suppose that multiplying $\hat{A}$ by j turns it vertically upwards.

Now, having said what j does, what *is* it? Well, that can be easily settled. If multiplying $\hat{A}$ by j turns it vertically upwards or through a right angle in a counter-clockwise direction as shown by the arrow, then multiplying $j\hat{A}$ by j again should turn it through another right angle in the same direction, or to two right angles from its first direction, which gives us $-\hat{A}$, or $-1\hat{A}$. Hence $j \times j \times \hat{A} = -1\hat{A}$, and since this must always be true whatever the vector $\hat{A}$ may be, we find that $j \times j = -1$. But according to ordinary algebraic rules, $j \times j = j^2$, hence $j^2 = -1$, and therefore $j = \sqrt{-1}$. Consequently, having decided what j was to do, we have now found what it is. Once agree that there is a quantity j which turns a vector through a right angle, then j must be a multiplier which, when used twice, turns it through two right angles, or is equivalent to multiplying it by -1, and therefore j must be equivalent to $\sqrt{-1}$, which when repeated twice, *i.e.* $\sqrt{-1} \times \sqrt{-1}$, becomes -1.

Now, to those whose mathematics have been confined to ordinary algebra and trigonometry this result is absurd. It is expressly stated in the majority of books on algebra that the square root of a negative quantity cannot exist, because the square of a positive quantity and of a negative quantity are both positive. The difficulty lies in the fact that algebra only deals with positive and negative quantities; and, as we have seen, if a positive quantity is represented by a line or vector, a negative one is represented by a vector in the opposite direction but in the same straight line with the first. Hence, so long as we are dealing with ordinary positive and negative signs in algebra we are really concerned geometrically with quantities all along the same line, and we are incapable of imagining anything outside that line. Any perfectly real geometrical line in another direction is therefore *imaginary* in ordinary algebra if we are confined to positive and negative quantities only, and that is why $\sqrt{-1}$ is called an imaginary quantity in algebra, although it is perfectly real geometrically. Ordinary algebra, in fact, when considered from the geometrical point of view, may be considered as mathematics in a pipe or along a track as in our generator and motor problem in the last chapter, where one can go forwards or backwards but never to the side, and in which everything outside the pipe or track is invisible or imaginary. Of course, this is perfectly well recognised in many of the higher, and in some of the more elementary, text-books on algebra, but it is unfortunate that it is not generally clearly emphasised, and that therefore students come to the idea of $\sqrt{-1}$ in geometry as something transcendental and as a highly artificial conception. Such a notion is just as absurd as to say that north and south are real, but that east and west are imaginary directions. It is to be hoped, therefore, that students will recognise that while $\sqrt{1}$ is either ± 1 or a unit distance forwards or backwards, $\sqrt{-1}$ is a unit distance sideways, and is just as real as the others.

We are now agreed that $j = \sqrt{-1}$ is a multiplier that turns a vector round through a right angle in a counter-clockwise direction. Hence, starting with the horizontal vector $\hat{A}$ and multiplying it successively by j, we first have $j\hat{A}$ vertically upwards, $j^2A = -\hat{A}$ horizontally backwards, $j^3A = j \times -1\hat{A} = -jA$ vertically downwards (as is correct, since it is opposite in sign to jA, which was vertically upwards), and $j^4\hat{A} = -1 \times -1\hat{A} = \hat{A}$, or back to where we were at first, as the j^4 has turned it through four right angles, or one complete revolution. Thus we have the simple relations—

j	j^2	j^3	j^4	j^5	j^6	j^7	j^8
j	-1	$-j$	$+1$	j	-1	$-j$	$+1$

Also $\frac{1}{j} = -\frac{j^2}{j} = -j$, since $j^2 = -1$. Although it is not necessary to have a name for j, it is sometimes convenient to refer to it by name. In the language of quaternions, with which we are really dealing here, a quantity which turns a vector round is called a versor (as in the word reverse = re-verse = turn back, or transverse = turn across), and since j is the multiplier which turns through a right angle or quadrant it is called a *quadrantal versor.*

Transformation of Vector Expressions.—We can now return to our original vector of length q and inclination θ, or of components a and b, and write it in either one of two forms, viz. $q\angle\theta$ or $a + jb$. Since these two forms mean the same thing, we can convert from one to the other. From the diagram we have evidently $\frac{a}{q} = \cos\theta$, or

$$a = q\cos\theta \quad \text{and} \quad b = q\sin\theta \quad . \quad . \quad . \quad (38)$$

which enables us to convert from the first form to the second;

also $$q = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan\theta = \frac{b}{a} \quad . \quad . \quad (39)$$

which gives the mode of conversion from the second to the first. Both of these conversions are most useful, as will be seen later.

Vector or Quaternion Multiplication.—But now we come to a most interesting and important matter, which again requires a little imagination to follow it. Either of the expressions $q\angle\theta$ or $a + jb$ represents the vector of length q and inclination θ, or of horizontal and vertical components a and b. Let the vector be an impedance of resistance r and reactance x. The impedance may then be written as $\hat{\mathrm{I}} = \sqrt{r^2 + x^2}\angle\phi$, where $\tan\phi = \frac{x}{r}$, or as $r + jx$. But now suppose that a current C passes through the coil, and we know that the P.D. $\hat{\mathrm{V}} = \hat{\mathrm{I}}\hat{\mathrm{C}}$, or is got by multiplying the current by the impedance. If we start with a vector diagram (Fig. 122) in which the current is represented as horizontal, then it is simply $\bar{\mathrm{C}}$ and $\hat{\mathrm{V}} = \hat{\mathrm{I}}\bar{\mathrm{C}} = \sqrt{r^2 + x^2}\,.\,\bar{\mathrm{C}}\angle\phi$, or $= r\bar{\mathrm{C}} + jx\bar{\mathrm{C}}$, which gives us the P.D. as a voltage $\bar{\mathrm{C}}\mathrm{I}$ in phase with the impedance or leading by the same angle ϕ in front of the current, and having components of $r\bar{\mathrm{C}}$ and $x\bar{\mathrm{C}}$, the resistance and reactance components respectively, as we know should be the case. There is nothing new or remarkable about this.

But the current $\bar{C}$ need not be the horizontal line in the diagram. If we have a coil of impedance $\hat{I} = I \angle \theta$ or $r + jx$, we can pass a current through it of any magnitude and any phase, if we control it by external means. The current $\bar{C}$ must then be written as a vector either in the form $\bar{C} \angle \theta$ or $\bar{C}_1 + j\bar{C}_2$, where $\bar{C} = \sqrt{\bar{C}_1^2 + \bar{C}_2^2}$ and $\tan \theta = \frac{\bar{C}_2}{\bar{C}_1}$. But now when finding the P.D. we have $\hat{V} = \hat{I}\hat{C} =$ either $I \angle \phi \times \bar{C} \angle \theta$ or $(r + jx)(\bar{C}_1 + j\bar{C}_2)$, *i.e.* one of our vectors has to be multiplied by the other.

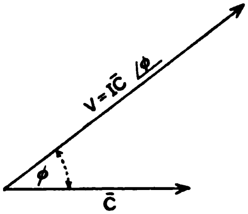

FIG. 122.—Product of $I \angle \phi$ and $\bar{C}$.

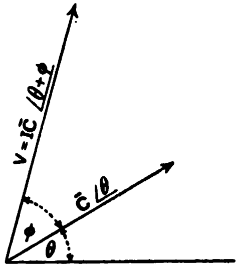

FIG. 123. — Product of $I \angle \phi$ and $\bar{C} \angle \theta$.

There is no difficulty in seeing what must be the result. If we have a current of strength $\bar{C}$ amperes passing through an impedance of magnitude I ohms, the magnitude of the P.D. must be $I\bar{C}$ volts. Further, the P.D. must lead in front of the current by the angle ϕ, by which the impedance leads in front of the resistance. But the current vector has already a lead or inclination to the horizontal of θ; consequently the P.D. must lead beyond this by ϕ, or have a total lead or inclination of $\theta + \phi$. Hence our result (Fig. 123) is that the P.D. $\hat{V} = \hat{I}\bar{C} = I \angle \theta \times \bar{C} \angle \phi = I\bar{C} \angle (\theta + \phi)$. Hence the product of the two vectors is a vector the magnitude of which is got by multiplying the magnitudes of the original vectors together, and the lead of which is the sum of the original angles of lead.

But now see what has happened. We started with the original vector $\bar{C} \angle \phi$ and we multiplied it by $I \angle \theta$. The result was to increase or stretch the original vector from $\bar{C}$ to $\bar{C}I$, and to turn it round from an inclination of ϕ to an inclination of $\phi + \theta$, or through an angle ϕ. The quantity $\hat{I} = I \angle \theta$ therefore, when used as a multiplier to the vector $\hat{C}$, performed the operation of stretching it I times and turning it through the angle θ. A multiplier which stretches and turns a vector is called a *quaternion*, which is made up of two factors, a stretching or extending factor or *tensor* I and a turning factor or *versor* $\angle\theta$. Hence the quantity $\hat{I}$ or $I\angle\theta$ if standing alone represents a vector of length I inclined at an angle of θ with the ~~vertical~~, H but if it stands as a multiplier in front of another vector it becomes a quaternion of tensor or stretching power I and of versor or turning power θ.

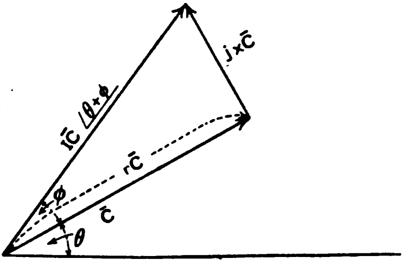

FIG. 124.—Product of $(r+jx)$ and $\bar{C}\angle\theta$.

The fact that the product of two quaternions or vectors like $I\angle\phi$ and $\bar{C}\angle\theta$ is $I\bar{C}\angle(\theta+\phi)$ can be easily shown by writing $\hat{I} = r + jx$ and $\hat{C} = \bar{C}_1 + j\bar{C}_2$, and it forms an instructive exercise in the handling of these vector quantities (Fig. 124). The product of these two vectors is $(r+jx)(\bar{C}_1 + j\bar{C}_2)$, which when multiplied out in the ordinary way gives

$$r\bar{C}_1 + rj\bar{C}_2 + jx\bar{C}_1 + j^2x\bar{C}_2$$

and since $j^2 = -1$ we have $r\bar{C}_1 - x\bar{C}_2 + j(x\bar{C}_1 + r\bar{C})$. Hence the result should be a vector of which the horizontal component is $a = r\bar{C}_1 - x\bar{C}_2$, and the vertical component is $b = x\bar{C}_1 + r\bar{C}_2$. The magnitude of this result should be—

$$\begin{aligned}\sqrt{a^2+b^2} &= \sqrt{(r\bar{C}_1 - x\bar{C}_2)^2 + (x\bar{C}_1 + r\bar{C}_2)^2} \\ &= \sqrt{r^2\bar{C}_1^2 - 2rx\bar{C}_1\bar{C}_2 + x^2\bar{C}_2^2 + x^2\bar{C}_1^2 + 2rx\bar{C}_1\bar{C}_2 + r^2\bar{C}_2^2} \\ &= \sqrt{(r^2+x^2)(\bar{C}_1^2 + \bar{C}_2^2)}\end{aligned}$$

But $r^2 + x^2 = I^2$ and $\bar{C}_1^2 + \bar{C}_2^2 = C^2$, so that the magnitude of the product of $r + jx$ and $\bar{C}_1 + j\bar{C}_2$, $= \sqrt{I^2\bar{C}^2} = I\bar{C}$, as we expected. The angle which the final vector makes with the horizontal is given by $\tan\theta' = \dfrac{\text{vertical component}}{\text{horizontal component}} = \dfrac{x\bar{C}_1 + r\bar{C}_2}{r\bar{C}_1 - x\bar{C}_2}$, or dividing the numerator and denominator by $r\bar{C}_1$,

$$\tan\theta' = \frac{\dfrac{x}{r} + \dfrac{\bar{C}_2}{\bar{C}_1}}{1 - \dfrac{x}{r}\dfrac{\bar{C}}{\bar{C}_1}}$$

But in the original vectors $\tan\theta = \frac{\bar{C}_2}{\bar{C}_1}$ and $\tan\phi = \frac{x}{r}$, hence we have $\tan\theta' = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}$ and, as is shown in any work on trigonometry, this is equal to $\tan(\theta + \phi)$. Thus $\tan\theta' = \tan(\theta + \phi)$, or $\theta' = \theta + \phi$, as before stated.

It is now quite easy to make calculations concerning any simple harmonically varying quantities by vector methods without any need for diagrams.

If we briefly review our back work, we shall see that we first obtained the mechanical relations $F_1 = \frac{1}{Y}s$, $F_2 = fv$, $F_3 = ma$, and the corresponding electrical relations $V_1 = \frac{1}{K}Q$, $V_2 = rC$, and $V_3 = LA$; after which we showed that the rate of change of a simple harmonic quantity was obtained by multiplying it by $p = 2\pi n$ and making it lead 90°, that simple harmonic quantities could be represented and compounded like vectors or cranks, and that the effective value of a S.H.M. was its maximum value divided by $\sqrt{2}$, and could also be expressed as a vector. First taking the fact that the rate of change of a S.H.M. is obtained by multiplying it by p and making it lead 90°, this is simply equivalent to multiplying it by jp, and we therefore have the simple rule:

The rate of change of any simple harmonic quantity of frequency n periods per sec. is obtained by multiplying it by jp, where $j = \sqrt{-1}$, *and* $p = 2\pi n$.

Hence if we have our piston moving with a S.H.M. of maximum displacement $\hat{s}$ we must have v = rate of change of s, or $\hat{v} = jp\hat{s}$, and similarly a = rate of change of v, or $\hat{a} = jp\hat{v}$. In like manner for an electrical oscillation $\hat{C} = jp\hat{Q}$, and $\hat{A} = jp\hat{C}$. We have also $\hat{a} = jp\hat{v} = j^2p^2\hat{s}$, since $\hat{v} = jp\hat{s}$; and since $j^2 = -1$, $\hat{a} = -p^2\hat{s}$, as we have frequently shown previously. Again, since $\hat{v} = jp\hat{s}$ we have $\hat{s} = \frac{\hat{v}}{jp} = \frac{1}{j}\frac{\hat{v}}{p} = -\frac{j}{p}\hat{v}$, since $\frac{1}{j} = -j$, and similarly $Q = -\frac{j}{p}\hat{C}$.

We are therefore able to write down the relations for the passage of simple harmonic alternate currents through condensers, resistances, and inductances at once as follows:

Condenser.	$V_1 = \frac{1}{K}Q$	$\bar{Q} = -\frac{j}{p}\bar{C}$	$\bar{V}_1 = -\frac{j}{Kp}\bar{C}$	$\hat{V}_1 = -\frac{j}{Kp}\hat{C}$	$\hat{C}_1 = jKp\hat{V}$
Resistance.	$V_2 = rC$		$\bar{V}_2 = r\bar{C}$	$\hat{V}_2 = r\hat{C}$	$\hat{C}_2 = \frac{\hat{V}}{r}$
Inductance.	$V_3 = LA$	$\bar{A} = jp\bar{C}$	$\bar{V}_3 = jLp\bar{C}$	$\hat{V}_3 = jLp\hat{C}$	$\hat{C}_3 = -j\frac{\hat{V}}{Lp}$

where j means $\angle 90°$ or leading 90° and $-j$ means $\urcorner 90°$ or lagging 90°. The use of the vector effective values $\hat{V}$ and $\hat{C}$ in the last two columns is justified by the relations $\bar{V} = \frac{\hat{V}}{\sqrt{2}}$ and $\bar{C} = \frac{\hat{C}}{\sqrt{2}}$.

By vector effective values we mean vectors having a magnitude or length equal to the effective value of the quantity considered. These vectors will in what follows be represented by a circumflex ^ over the letter.

Turning next to combinations of capacity, resistance, and inductance in series, we have $\hat{V}$ equal to the vector sum of $\hat{V}_1$, $\hat{V}_2$, and $\hat{V}_3$, or—

$$\hat{V} = -\frac{j}{Kp}\hat{C} + r\hat{C} + jLp\hat{C} = \left\{r + j\left(Lp - \frac{1}{Kp}\right)\right\}\hat{C}$$

If we call $Lp - \frac{1}{Kp}$ the reactance x we have $\hat{V} = (r + jx)\hat{C} = \hat{I}\hat{C}$ where $\hat{I}$ is the P.D. when $\bar{C}$ is 1 and is termed the impedance of the circuit. Hence $\hat{I} = r + jx$ and $\tan\phi = \frac{\text{vertical component}}{\text{horizontal component}} = \frac{x}{r}$. This is exactly what we proved in Chapter X.

For circuits in parallel we may have any number of branches, each having a resistance r_1, r_2, &c., and a reactance $Lp - \frac{1}{Kp}$ of x_1, x_2, &c. Taking a single circuit first of impedance $\hat{I} = r + jx$, we have $\hat{V} = \mathrm{I}\hat{C}$, or $\hat{C} = \frac{\hat{V}}{\mathrm{I}} = \frac{\hat{V}}{r + jx}$. Now if two quantities $a + b$ and $a - b$ are multiplied together, we obtain $a^2 - b^2$ or the difference of their squares. Consequently, if we have a quantity $r + jx$ and multiply it by $r - jx$, we must have $r^2 - j^2x^2$, and since $j^2 = -1$, this becomes $r^2 + x^2$. Hence, if we take the relation $\hat{C} = \dfrac{\hat{V}}{r + jx}$ and multiply it both above and below by $r - jx$ we have—

$$\hat{C} = \hat{V}\frac{r - jx}{(r + jx)(r - jx)} = \hat{V}\frac{r - jx}{r^2 + x^2} = \hat{V}\left\{\frac{r}{r^2 + x^2} - j\frac{x}{r^2 + x^2}\right\}$$

If $\hat{V} = 1$, the current is $\hat{Y} = \dfrac{r}{r^2 + x^2} - j\dfrac{x}{r^2 + x^2}$ or $\dfrac{r}{r^2 + x^2}$ in phase with $\hat{V}$ and $\dfrac{x}{r^2 + x^2}$ lagging 90°. But the current in phase for one volt P.D. was what we called on p. 159 the conductance g of the circuit; and the current lagging 90° behind the unit voltage we called the susceptance b. Hence we arrive at the result that $g = \dfrac{r}{r^2 + x^2}$ and $b = \dfrac{x}{r^2 + x^2}$, as in (34) and (35), and the total current for one volt or admittance $\hat{Y} = \dfrac{1}{\hat{I}} = g - jb$, from which the magnitude of $\hat{Y}$ is the square root of the sum of the squares of the horizontal and vertical components, or $\sqrt{g^2 + b^2}$. Similarly, if we are given the admittance $\hat{Y} = g - jb$ we can find the impedance, resistance, and reactance by writing

$$\hat{I} = \frac{1}{\hat{Y}} = \frac{1}{g - jb} = \frac{g + jb}{(g + jb)(g - jb)} = \frac{g + jb}{g^2 + b^2}$$

$$= \frac{g}{g^2 + b^2} + j\frac{b}{g^2 + b^2} = r + jx$$

where $r = \dfrac{g}{g^2 + b^2}$ and $x = \dfrac{b}{g^2 + b^2}$. All these results were obtained before in Chapter XI, but the symbolic method is much quicker and saves the necessity for diagrams.

Vector Expression for Power.—It will be seen that if we employ this symbolic method of obtaining the P.D.'s and currents in various parts of a system of circuits, each can be expressed eventually in the form $\hat{V} = \bar{V}_x + j\bar{V}_y$ and $\hat{C} = \bar{C}_x + j\bar{C}_y$, $\bar{V}_x$ and $\bar{V}_y$ being the horizontal and vertical components of the effective

P.D., and $\bar{C}_x$ and $\bar{C}_y$ those of the current. Hence it follows that $\bar{V}_x$ and $\bar{C}_x$ are in the same direction or in phase with one another, and $\bar{V}_y$ and $\bar{C}_y$ are similarly in phase, while $\bar{C}_y$ is in quadrature with $\bar{V}_x$ and $\bar{C}_x$ in quadrature with $\bar{V}_y$. Hence, remembering that $\mathfrak{w} = \bar{V}\bar{C}$ when $\bar{V}$ and $\bar{C}$ are in phase, and zero when they are in quadrature, we have $\mathfrak{w} = \bar{V}_x\bar{C}_x + \bar{V}_y\bar{C}_y$ as the power in this case. Of course, if we know the current $\hat{C}$ and the impedance $\hat{I} = r + jx$, we have $\hat{V} = \hat{I}\hat{C}$ and $\mathfrak{w} = \bar{V}\bar{C}\cos\phi =$

$$\sqrt{r^2 + x^2}\,\bar{C}^2 \frac{r}{\sqrt{r^2 + x^2}} = \bar{C}^2 r,$$ as in Chapter XII.

Equivalent Resistance and Reactance of two Choking Coils in Parallel.—As an instance of the readiness with which general formulæ can be obtained by the symbolic method, let us take the case of two inductance coils of resistances r_a and r_b, and inductances L_a and L_b in parallel on an alternate-current circuit having a simple harmonic P.D. of frequency n.

Then we have $x_a = L_a p$ and $x_b = L_b p$, where $p = 2\pi n$ and $\hat{I}_a = r_a + jx_a$ and $\hat{I}_b = r_b + jx_b$. The total impedance, if they were in series, would be $\hat{I}_s = \hat{I}_a + \hat{I}_b = r_a + r_b + j(x_a + x_b)$. In parallel, however, we have—

$$\hat{Y}_a = \frac{1}{\hat{I}_a} = \frac{1}{r_a + jx_a} = \frac{r_a}{r_a^2 + x_a^2} - j\frac{x_a}{r_a^2 + x_a^2} = \frac{r_a}{I_a^2} - j\frac{x_a}{I_a^2}$$

and $$\hat{Y}_b = \frac{1}{\hat{I}_b} = \frac{1}{r_b + jx_b} = \frac{r_b}{r_b^2 + x_b^2} - j\frac{x_b}{r_b^2 + x_b^2} = \frac{r_b}{I_b^2} - j\frac{x_b}{I_b^2}$$

from which the total admittance—

$$\hat{Y}_p = \hat{Y}_a + \hat{Y}_b = \frac{r_a}{I_a^2} + \frac{r_b}{I_b^2} - j\left(\frac{x_a}{I_a^2} + \frac{x_b}{I_b^2}\right)$$

The resultant impedance in parallel is then—

$$\hat{I}_p = r_p + jx_p = \frac{1}{\hat{Y}_p} = \frac{1}{\frac{r_a}{I_a^2} + \frac{r_b}{I_b^2} - j\left(\frac{x_a}{I_a^2} + \frac{x_b}{I_b^2}\right)}$$

$$= \frac{\frac{r_a}{I_a^2} + \frac{r_b}{I_b^2} + j\left(\frac{x_a}{I_a^2} + \frac{x_b}{I_b^2}\right)}{\left(\frac{r_a}{I_a^2} + \frac{r_b}{I_b^2}\right)^2 + \left(\frac{x_a}{I_a^2} + \frac{x_b}{I_b^2}\right)^2}$$

Hence the equivalent resistance of the combination is—

$$r_p = \frac{\frac{r_a}{I_a^2} + \frac{r_b}{I_b^2}}{\left(\frac{r_a}{I_a^2} + \frac{r_b}{I_b^2}\right)^2 + \left(\frac{x_a}{I_a^2} + \frac{x_b}{I_b}\right)^2}$$

and the equivalent reactance—

$$x_p = \frac{\frac{x_a}{I_a^2} + \frac{x_b}{I_b^2}}{\left(\frac{r_a}{I_a^2} + \frac{r_b}{I_b^2}\right)^2 + \left(\frac{x_a}{I_a^2} + \frac{x_b}{I_b^2}\right)^2}$$

both of which quantities are dependent on the frequency. If the coils had no inductance, $x_a = x_b = 0$, $I_a = r_a$ and $I_b = r_b$, and we have—

$$r_p = \frac{\frac{1}{r_a} + \frac{1}{r_b}}{\left(\frac{1}{r_a} + \frac{1}{r_b}\right)^2} = \frac{1}{\frac{1}{r_a} + \frac{1}{r_b}}$$

which is the ordinary result for parallel resistances. If, on the other hand, the coils had inductance but negligible resistance,

$$I_a = x_a \quad \text{and} \quad I_b = x_b$$

$$r_p = 0 \quad \text{and} \quad x_p = \frac{1}{\frac{1}{x_a} + \frac{1}{x_b}}$$

Shunted Condenser.—If a condenser of capacity K is shunted with a non-inductive resistance r, the conductance of the system $g = \frac{1}{r}$ and susceptance $b = -Kp$, and the admittance of the combination $\hat{Y} = g - jb = \frac{1}{r} + jKp = \frac{1 + jKrp}{r}$. Hence the impedance of the combination—

$$\hat{I} = \frac{1}{\hat{Y}} = \frac{r}{1 + jKrp} = \frac{r(1 - jKrp)}{1 + K^2r^2p^2} = \frac{r}{1 + K^2r^2p^2} - j\frac{Kr^2}{1 + K^2r^2p^2}p$$

Comparing this with the formula for the impedance of a choking coil, viz. $\hat{I} = R + jX = R + jLp$, we see that R corresponds to $\frac{r}{1 + K^2r^2p^2}$ and L to $-\frac{Kr^2}{1 + Kr^2p^2}$. If Krp is small in comparison with 1, as would most frequently be the case, $K^2r^2p^2$ may be neglected in the denominator, and $R = r$ and $L = -Kr^2$, or the shunted condenser is equivalent to a choking coil having a resistance of the same value as that of the shunt, and a *negative* inductance of Kr^2, which is, of course, impossible with an ordinary coil. But this means that if we put a condenser K shunted with a resistance r_1 in series with a coil of inductance L, as in Fig. 125, there will be no resultant inductance if $Kr^2 = L$; hence the impedance will only be the total resistance $r_1 + r_2$, and the current will be in phase with the P.D. This is like the result which we obtained from a condenser and choking coil in series on p. 135, when $Lp = \frac{1}{Kp}$, but in that case the compensation of inductance by capacity was only correct for one particular frequency, while here it is true for all frequencies if $Kr^2 = L$, and Krp is small compared with unity. This mode of compensation for inductance is therefore very useful under some circumstances.

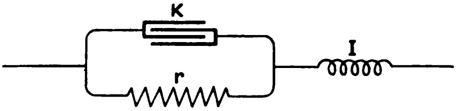

FIG. 125.—Impedance and shunted condenser in series.

The general expression for the impedance of the combination in Fig. 125 is evidently

$$\hat{I} = \frac{r_1}{1 + K^2r_1^2p^2} + r_2 + j\left\{L - \frac{Kr_1^2}{1 + K^2r_1^2p^2}\right\}p$$

These examples should be enough to show how useful vector methods of calculation are in connection with alternate-current problems. As a conclusion to this chapter we have collected the chief alternate-current formulæ expressed in the symbolic form in Table XVII, which should be compared with Table XVI at the end of Chapter XI.

TABLE XVII.—ALTERNATE-CURRENT FORMULÆ IN SYMBOLIC OR VECTOR FORM

—	Circuits in Series.	Circuits in Parallel.
Condenser Resistance Inductance	$\hat{V}_1 = -\frac{j}{Kp}\hat{C}$ $\hat{V}_2 = r\hat{C}$ $\hat{V}_3 = jLp\hat{C}$	$\hat{C}_1 = jKp\hat{V} = j\frac{\hat{V}}{\frac{1}{Kp}}$ $\hat{C}_2 = \frac{\hat{V}}{r}$ $\hat{C}_3 = -j\frac{\hat{V}}{Lp}$
Condenser Resistance and Inductance combined	$\hat{V} = \hat{V}_1 + \hat{V}_2 + \hat{V}_3 = \left\{r + j\left(Lp - \frac{1}{Kp}\right)\right\}\hat{C}$ $= (r + jx)\hat{C} = \hat{I}\hat{C} = \mathrm{I}\bar{C} \angle \phi$ where $\hat{I} = r + jx$, $\mathrm{I} = \sqrt{r^2 + x^2}$, and $x = Lp - \frac{1}{Kp}$ $\tan\phi = \frac{x}{r}$	$\hat{C} = \frac{1}{\hat{I}}\hat{V} = \hat{Y}\hat{V} = \frac{\bar{V}}{\mathrm{I}} \ulcorner \phi$ where $\hat{Y} = \frac{1}{\hat{I}} = \frac{1}{r + jx} = \frac{r - jx}{r^2 + x^2} = g - jb$ $g = \frac{r}{r^2 + x^2} = \frac{r}{\mathrm{I}^2}$, and $b = \frac{x}{r^2 + x^2} = \frac{x}{\mathrm{I}^2}$ $\tan\phi = \frac{b}{g}$
Compound Impedances combined	$\hat{I} = \hat{I}_a + \hat{I}_b + \hat{I}_c$, &c. $= \Sigma\hat{I}_a$ $R = \Sigma r_a$, $X = \Sigma x_a$ $\tan\phi = \frac{X}{R} = \frac{\Sigma x_a}{\Sigma r_a}$	$\hat{Y} = \hat{Y}_a + \hat{Y}_b + \hat{Y}_c$, &c. $= \Sigma\hat{Y}_a$ $G = \Sigma g_a$, $B = \Sigma_1 b_a$ $\tan\phi = \frac{B}{G} = \frac{\Sigma b_a}{\Sigma g_a}$
Equivalent Admittance or Impedance	$\hat{Y} = \frac{1}{\hat{I}} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = G + jB$ where $G = \frac{R}{R^2 + X^2} = \frac{R}{\mathrm{I}^2}$, and $B = \frac{x}{R^2 + X^2} = \frac{x}{\mathrm{I}^2}$ or $\hat{Y} = \mathrm{Y} \ulcorner \phi$, where $\mathrm{Y} = \sqrt{G^2 + B^2}$ and $\tan\phi = \frac{B}{G}$	$\hat{I} = \frac{\mathrm{I}}{\hat{Y}} = \frac{\mathrm{I}}{G - jB} = \frac{G + jB}{G^2 + B^2} = R + jX$ where $R = \frac{G}{G^2 + B^2}$, and $X = \frac{B}{G^2 + B^2}$ or $\hat{I} = \mathrm{I} \angle \phi$, where $\mathrm{I} = \sqrt{R^2 + X^2}$ and $\tan\phi = \frac{X}{R}$
Power	If $\hat{V} = \bar{V}_x + j\bar{V}_y$ and $\hat{C} = \bar{C} + j\bar{C}_y$ $\bar{w} = \bar{V}_x\bar{C}_x + \bar{V}_y\bar{C}_y = \bar{V}\bar{C}\cos\phi$	

PART IV

MUTUAL AND EXTERNAL ACTIONS

CHAPTER XIV

MUTUAL INDUCTION

HITHERTO we have been dealing with our electrical circuits and mechanical analogies as if capacity, resistance, and inductance, or their corresponding mechanical quantities, elasticity, friction, and inertia, were the only influences which controlled the current or motion. This is perfectly true so long as we have a circuit which is not in the neighbourhood of any other circuits, or a body whose motion is not affected by any other moving body. But when two circuits are close together there are some new effects which have their corresponding mechanical analogies, and these we proceed to consider, as they are of great importance both from the theoretical and practical standpoint.

The fundamental action which takes place between two circuits was discovered by Faraday about the same time as his discovery of the induction of a current by the introduction of a magnet into a coil of wire.* If instead of a moving magnet he used a fixed iron core which could be magnetised by passing a current round a second coil quite independent of the first, he found that each time a current was made or broken in the magnetising coil, a deflection was produced on a galvanometer connected to the other coil, the deflections being only momentary in each case, and being in the opposite direction at break to that at make, as the currents induced by the motion of a magnet in a coil were opposite in direction at the introduction and withdrawal of the magnet. This is, of course, quite natural, as the making of the current magnetises the core, and is equivalent to introducing a magnet into the coil, while the breaking of the current causes the magnetisation to disappear as if the magnet were withdrawn.

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* This discovery was also made independently by Henry in America, whence the name for the unit of inductance.

Now if we consider two coils side by side, and an iron core through them, as in Fig. 126, either of the two coils can be used to magnetise the iron, and the other for observing the induced current. It is usual to term the magnetising coil the *primary* or *inducing* circuit, and the one in which the current is induced the *secondary* or *induced* circuit. In the diagram the upper circuit is connected to the battery and key and is, therefore, the primary circuit, while the lower coil forms the secondary, and is connected to the galvanometer. On depressing the key a current flows round the primary coil in the direction shown by the arrow, and produces a magnetic flux in the direction shown. This flux not only passes through the primary coil itself but the greater part of it passes also through the secondary coil, and induces an E.M.F. in both coils proportional to the rate of change of the flux. It is clear that both these E.M.F.'s must have the same direction round the core; and we remember that in Chapter II it was proved that the E.M.F. in the primary coil tended to stop the change in the current. If the current, then, is in the direction shown by the arrow and increasing in strength, the induced E.M.F. must tend to stop this increase and therefore be in a direction opposite to the arrow. The secondary E.M.F. will be in the same direction, and will therefore be opposite in direction to the primary current when it is increasing. On the other hand, if

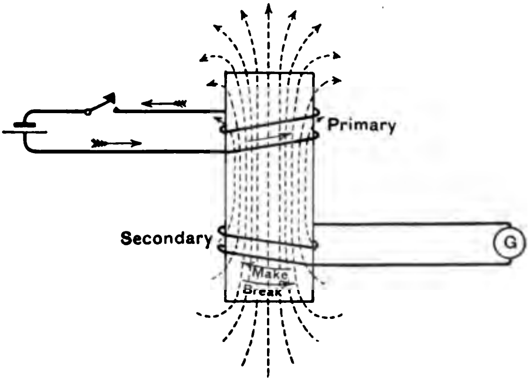

FIG. 126.—Mutual induction.

the key is released, and the primary current stops, the dying away of the flux produces an E.M.F. tending to make the current run on, and therefore in the direction of the arrow; the secondary E.M.F. will consequently be in this direction. If the secondary coil is connected to a galvanometer, the only E.M.F. in the secondary circuit is that induced in it by the core; and the secondary current will then be opposite to the primary current at make, and in the same direction as the primary current at break, which is what is actually observed. The secondary coil, in fact, forms a method of separating out the inductance or inertia effects in the primary from the driving force.

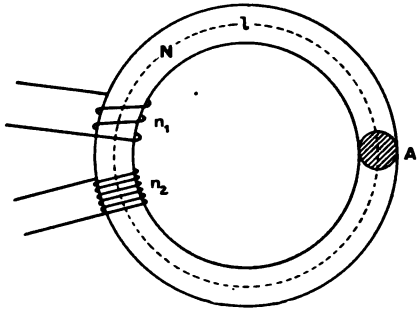

FIG. 127.—Coils with closed magnetic circuit.

It is clear that since the secondary E.M.F., like the primary induced E.M.F., is proportional to the rate of change of the flux, it is proportional to the rate of change of the primary current, if the flux is proportional to the current. The secondary E.M.F. in volts, like the primary E.M.F., is equal to the number of hundreds of millions of linkages per second. Now suppose that when the current in the primary is increased by one ampere it causes M hundreds of millions of linkages to be coupled with the secondary coil (the coupling of linkages being reckoned as number of extra lines × number of secondary turns, as on p. 19). Then if the current in the primary increases or accelerates at the rate of one ampere per second, the linkages with the secondary coil increase at the rate of M hundreds of millions per second, and there is therefore a secondary E.M.F. of M volts induced in it. If the primary current accelerates at the rate of A_1 amperes per second, the secondary E.M.F. in volts will then be $E_2 = -MA_1$, just as the primary voltage to overcome self-induction was LA_1 where L was the number of hundreds of millions of linkages coupled with the primary coil itself for a variation of current of one ampere. The negative sign for E_2 is used because the E.M.F. is in the opposite direction to the primary current when it is increasing, or its acceleration is positive. The E.M.F. LA_1 induced in the primary is also in the same direction or negative, but in the equation $V_3 = LA$

we took the voltage necessary to *overcome* this back E.M.F., which was, of course, in the opposite direction.

The quantity M is called the coefficient of *mutual* induction or the *mutual* inductance between the primary and secondary circuits, to distinguish it from the coefficient of *self*-induction or simple inductance of a single circuit. Since, like self-induction, it is represented by a number of lines of force cut per ampere, it is measured in henries, and we have the following definition:

The coefficient of mutual induction between one circuit and another in henries is the number of hundreds of millions of linkages coupled with the second circuit when the current in the first is increased by one ampere.

We also have the important rule:

When two circuits have a coefficient of mutual induction of M henries between them, and the current varies in one of them, an E.M.F., E_2 is induced in the other such that

$$E_2 = -MA_1 \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Similarly, if the current in the second coil varies, an E.M.F., E_1 is induced in the first such that

$$E_1 = -MA_2 \quad . \quad . \quad . \quad . \quad . \quad (41)$$

It is quite evident that if a current in the first coil induces an E.M.F. in the second, a current in the second must induce an E.M.F. in the first, since both coils are round the same core. But it is not at first evident that the coefficient of mutual induction from the first to the second is the same as from the second to the first. It is, however, easy to see this in the case of two coils side by side on a ring of iron, as in Fig. 127. For in this case we know that the magnetising force due to a coil is $\frac{4\pi}{10}Cn$, and the reluctance of the circuit is $\mathsf{R} = \frac{l}{A\mu}$ where l is the length of magnetic path, A the area of section of the iron, and μ the permeability. If the permeability is constant, which is only true for air or for iron at moderate induction densities, the reluctance R is constant, and the magnetic flux N through the iron is $\frac{4\pi}{10}\frac{Cn}{\mathsf{R}}$, which is proportional to C. The magnetic flux in lines for one ampere is therefore $\frac{N}{C} = \frac{4\pi}{10\mathsf{R}}n$, and in hundreds of millions of lines for one ampere $= \frac{4\pi}{10^8 \times 10\mathsf{R}} n = \frac{4\pi}{10^9\mathsf{R}} n$. Now if the primary and secondary coils have n_1 and n_2 turns respectively, the flux in hundreds of millions of lines per ampere in the primary coil $= \frac{4\pi}{10^9\mathsf{R}} n_1$, and the flux cut by the secondary of n_2 turns $= \frac{4\pi}{10^9\mathsf{R}} n_1 n_2$ in hundreds of millions of linkages per ampere of primary current, which is accordingly the coefficient of mutual induction in henries between the primary and secondary circuits.

But if we had started with the secondary coil and used it as a primary, we should simply have had a magnetising coil of n_2 instead of n_1 turns and the flux per ampere $= \frac{4\pi}{10^9\mathsf{R}} n_2$. The number of linkages coupled with the original primary coil, which is now the secondary coil of n_1 turns, will therefore be $\frac{4\pi}{10^9\mathsf{R}} n_2 n_1$ hundreds of millions of linkages per ampere, which is the same as before.

Hence, whenever two coils are wound over the same iron circuit, of which the reluctance R is constant, the coefficient of mutual induction

$$M = \frac{4\pi}{10^9\mathsf{R}} n_1 n_2 \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and is the same whichever coil is primary or secondary. The equality is true whether they are on the same iron core or not, but this is more difficult to prove.

It should be noticed that since the flux due to the primary coil per ampere $= \frac{4\pi}{10^9\mathsf{R}} n_1$, the flux cut in hundreds of millions of linkages per ampere by the primary coil itself is this flux multiplied by the number of turns in the primary coil itself, or n_1.

Hence—

$$L_1 = \frac{4\pi}{10^9\mathsf{R}} n_1^2 \quad . \quad . \quad . \quad . \quad . \quad (43)$$

henries by the definition of the primary inductance, and similarly the inductance of the secondary itself is

$$L_2 = \frac{4\pi}{10^9 \mathsf{R}} n_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

henries.

If we multiply these last two results together we have $L_1 L_2 = \left(\frac{4\pi}{10^9 \mathsf{R}}\right)^2 n_1^2 n_2^2$, and if we square the coefficient of mutual induction M we have $M^2 = \left(\frac{4\pi}{10^9 \mathsf{R}}\right)^2 n_1^2 n_2^2$, which is the same as $L_1 L_2$. Hence, when two coils are close together on the same iron core so that the whole of the flux produced by the one passes through the other—

$$M^2 = L_1 L_2 \text{ or } M = \sqrt{L_1 L_2} \quad . \quad . \quad . \quad (45)$$

It is worth noting that if both coils have the same number of turns, or $n_1 = n_2$, n_1^2, n_2^2, and $n_1 n_2$ are all equal, whence $L_1 = L_2 = M$. When, however, the two coils are further apart, as in Fig. 128, and especially if there is not a good magnetic circuit, each of the coils, besides sending flux through the other, may produce a flux which passes through itself but leaks out of the core and does not cut the secondary. In this case it is clear that while all the lines contribute to the self-induction of the coils, only the flux common to both produces the mutual induction and M is less than $\sqrt{L_1 L_2}$. As the magnetic circuit gets worse and the distance between the circuits greater, M gets less and less and finally becomes practically zero. The ratio of

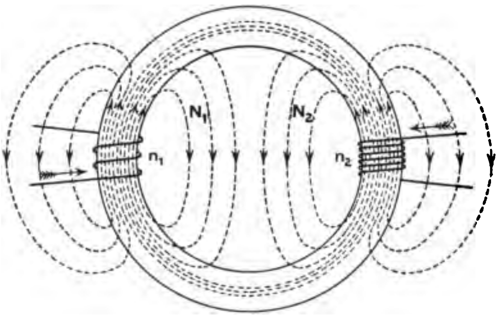

FIG. 128.—Coils with magnetic leakage.

M to $\sqrt{L_1L_2}$ is now spoken of in wireless telegraphic work as the *coefficient of coupling*. If the coefficient of coupling is greater than ·5 the circuits are said to be *close* coupled, while if it is less than ·5 the coupling is said to be *loose*.

Equations for Two Mutually Inducing Circuits.—Suppose, as in Fig. 129, we have two circuits each made up of a battery or generator with capacity and resistance, and let them be wound on the iron ring so as to have self and mutual induction. Then we know that the voltage required to charge the condenser is $V = \frac{1}{K_1} Q_1$, to force the current through the resistance $V_2 = r_1C_1$, and to accelerate it through the inductance $= L_1A_1$. Hence we must have $\frac{1}{K_1} Q_1 + r_1C_1 + L_1A_1 = V_1 +$ the E.M.F. due to mutual induction from the second circuit $E_1 = - MA_2$.

Hence $\frac{1}{K_1} Q_1 + r_1C_1 + L_1A_1 = V_1 - MA_2$, and similarly for the second circuit $\frac{1}{K_2} Q_2 + r_2C_2 + L_2A_2 = V_2 - MA_1$, or—

$$\frac{1}{K_1} Q_1 + r_1C_1 + L_1A_1 + MA_2 = V_1 \quad . \quad . \quad (46)$$

$$\frac{1}{K_2} Q_2 + r_2C_2 + L_2A_2 + MA_1 = V_2 \quad . \quad . \quad (47)$$

which are the equations which we have to solve whenever circuits having resistance, inductance, and capacity act upon one another, as for example in wireless telegraphy.

We shall not attempt to solve these equations immediately, however, but we will call attention to one important case. Suppose that the first circuit consisted of battery or generator with resistance and condenser as in Fig. 129, but that the secondary coil was simply short-circuited on itself, and that it had practically no resistance. Then we should have $V_2 = 0$, as there is no secondary battery or generator $\frac{1}{K_2} = 0$, $r_2 = 0$, and our equations are—

$$\frac{1}{K_1} Q_1 + r_1C_1 + L_1A_1 + MA_2 = V_1$$

$$L_2A_2 + MA_1 = 0$$

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But the second of these equations says that $L_2A_2 = -MA_1$ from which $A_2 = -\frac{M}{L_2}A_1$ and $MA_2 = -\frac{M^2}{L_2}A_1$. Hence, putting this for the last term on the left-hand side of the first equation, we have—

$$\frac{1}{K_1}Q_1 + r_1C_1 + L_1A_1 - \frac{M^2}{L_2}A_1 = V_1$$

or
$$\frac{1}{K_1}Q_1 + r_1C_1 + \left(L_1 - \frac{M^2}{L_2}\right)A_1 = V_1$$

Now this is an equation of just the same form as for a single circuit containing capacity, resistance, and inductance, except that we have $L_1 - \frac{M^2}{L_2}$ instead of L_1. In other words, the effect of the second coil is to reduce the self-induction of the first by the amount $\frac{M^2}{L_2}$. If the secondary is taken away so that $M = 0$, $\frac{M^2}{L_2} = 0$, and the primary inductance $= L_1$ as usual, but if the primary and secondary coils are wound close together upon a good iron circuit we saw just now that $M^2 = L_1L_2$, from which $\frac{M^2}{L_2} = L_1$ and $L_1 - \frac{M^2}{L_2} = 0$. The primary equation then becomes $\frac{1}{K_1}Q_1 + r_1C_1 = \cancel{0}\,V_1$ just as if there were no primary inductance at all. Hence mutual induction tends to reduce self-induction, and may practically eliminate it, if the coils are very 'close coupled,' *i.e.* wound close together, and the secondary coil is short-circuited.

The physical explanation of this is very simple; for we saw at the beginning of this chapter that when the primary current is increasing, the secondary current is in the opposite direction. But this means that as the primary is magnetising the core the secondary tends to demagnetise it, or to prevent the increase of the flux which gives the inductive back E.M.F.

in the primary. If the coils are close together on an iron core the whole of the flux which passes through the primary coil must also pass through the secondary, and if at the same time the secondary resistance is extremely small, the slightest cutting of lines of force, producing an E.M.F. however small, would produce a large demagnetising current. In fact, a coil of no resistance simply could not be penetrated by lines of force at all, as a current would always flow sufficient to keep them back. Of course, in practice the resistance of the coil makes the current die away before long, but the demagnetising currents reduce the primary induced E.M.F. If the secondary circuit is broken so that no current can flow, there will be no effect on the primary current, however large the mutual induction.

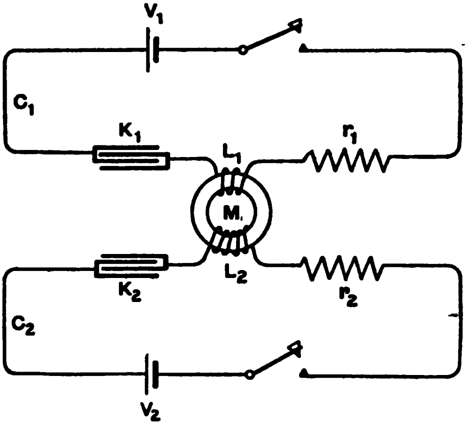

FIG. 129.—Mutually inducing circuits.

Mechanical Analogies to Mutual Induction.—In dealing with resistance, inductance, and capacity at the commencement of the volume, we proceeded from the mechanical to the electrical examples. We have started with the electrical case here because mechanical illustrations are not quite so familiar. However, there are a good many, and we now proceed to refer to a few of them.

First let us summarise the properties of mutual induction which we have just enumerated. They may be put in two statements: (*a*) When two circuits mutually affect one another, an electrical acceleration in either of them produces an electrical force in the other proportional to the acceleration and in the opposite direction. Or $E_2 = -MA_1$ and $E_1 = -MA_2$. (*b*) The mutually induced current in the secondary circuit has the effect of reducing the primary inductance or electrical inertia. Hence for a mechanical analogy to mutual induction between two moving bodies we must have the result that an acceleration of either body produces a force proportional to the acceleration tending to move the other in the opposite direction, and that the motion of the second body thus produced reduces the apparent inertia of the first body.

Before considering more exact analogies there are some fairly obvious mechanical examples which resemble mutual induction. Take the simple case of a horse and cart at rest on an ordinary road. Suppose that the horse suddenly pulls the cart with a definite steady force. This will cause the cart at first to accelerate in the direction of the force at a constant rate. But the horse cannot pull the cart forwards without producing a force backwards between his feet and the ground, as is shown by his feet slipping backwards if the ground is too smooth. Hence the horse cannot accelerate the cart forwards without producing a force tending to move the ground backwards, just as the acceleration of the primary current produced an opposite E.M.F. in the secondary. Again, if the horse has got the cart up to speed, and then suddenly stops pulling, the inertia of both the horse and the cart will tend to make them continue to move forwards, and will give a forward force against the road, like the forward secondary current on the stoppage of the primary current.

Anyone who has had experience of mats on polished wood floors ought to realise some of the properties of mutual induction in a most vivid manner. If a person standing on such a mat suddenly attempts to start forwards, *i.e.* to give himself a forward acceleration, the mat flies back, as the current in the secondary circuit did when there was little resistance or electrical friction. In fact, if there were no friction at all between the mat and the floor, any attempted motion would be instantly compensated by an opposite motion of the mat, and it would be absolutely impossible to move away from the same spot while on the mat, just as it was impossible to increase the flux in an iron core when there was a short-circuited secondary of negligible electrical friction or resistance.

On the other hand, in the practical case when there is a certain amount of friction between the mat and the floor, if a person is walking over it and suddenly stops, the mat slides forwards, like the forward secondary current at break.

Similarly, if we take an ordinary electric tramcar, motor car, or other self-propelled vehicle, and start it, a backward force is exerted on the rails or ground during the acceleration, while if the car is up to speed and the brakes are put on, the car tends to run on and to carry the track forwards with it.

These two analogies, although suggestive, are, however, imperfect, as even when the cart or tramcar is running at a

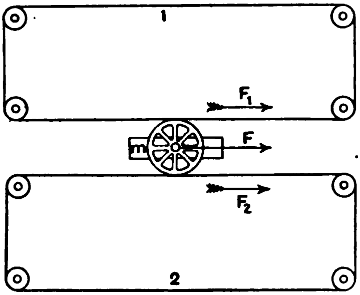

FIG. 130.—Mechanical model to illustrate mutual induction.

constant speed, there is a backward force against the ground owing to the pull or force necessary to overcome friction. Furthermore, force is required to accelerate the car, whether there is any recoil or not. In the case of the electric circuits there is no force whatsoever in the secondary when the primary current is steady, whatever the friction, and the mutual induction reduces the apparent inductance or inertia of the coil. We must therefore look a little further for a more exact analogy. A fairly clear and satisfactory mechanical analogy which fits in very well with our cord models is shown in Fig. 130. Here we have two of our cord circuits side by side, but instead of a heavy car in each circuit representing self-induction, we have a single car which runs along rails in between the circuits. Pivoted to this car is a horizontal wheel, and each cord takes a turn round it. The car and horizontal wheel are supposed to move without friction. Suppose first that cord 2 is clamped so as not to move, which is equivalent to breaking the secondary circuit. If the car were without mass or friction it would be evident that cord 1 can move in any manner without affecting cord 2. For the motion of cord 1 turns round the wheel, which will then roll on cord 2 and move the car along at half the speed of cord 1, but if the car has no friction or inertia it will offer no resistance to this motion and will communicate no force to the second cord. But if, on the other hand, we fix the car and unclamp cord 2, any motion of cord 1 will turn the wheel and give an equal and opposite movement to cord 2. Now, if instead of fixing the car we put a weight in it, it will resist any acceleration with a force which is communicated to both cords. If cord 1 moves with uniform speed and there is no friction in the car, there will be no resistance to the motion and no force communicated to cord 2. But if cord 1 accelerates forwards, the inertia of the car retards its motion, and the wheel tends to turn and imparts a backward force to cord 2. Consequently, if cord 2 is quite free and moves without any resistance whatever, the result of accelerating cord 1 will be that the car will not move, but that cord 2 will move backwards like the secondary current at make. Should, however, there be any resistance to the motion in cord 2, there will be a resultant force on the car which will cause it to accelerate; since it is supposed to have no friction, as soon as it has attained a speed equal to half that of cord 1 it will no longer offer any retarding force, and it will cease to communicate any force to cord 2, just as the E.M.F. of mutual induction ceases when the primary current is steady. When this stage has been reached and the motion of cord 1 is stopped, the car will tend to run on, and if cord 2 is free to move will carry it along with it, giving it a motion in the same direction as that of the original motion of cord 1.

If this arrangement is studied fully it will be seen that it reproduces practically all the phenomena of self and mutual induction in a pair of circuits wound on an iron core. If we denote the force with which the cord 1 tends to move the wheel and car by F_1, and that for cord 2 by F_2, while F is the force which the pulley communicates to the car, we have $F_1 + F_2 = F$, and $F_1 = F_2$ since the pulley can always turn freely; therefore $F_1 = F_2 = \frac{1}{2}F = \frac{1}{2}ma$ where m is the mass of the car and a its acceleration. It will be seen at once that the motion of the car is exactly midway between or the average of the motions of the two cords, from which $a = \dfrac{a_1 + a_2}{2}$. Hence

$$F_1 = F_2 = \tfrac{1}{2}ma = \tfrac{1}{4}m(a_1 + a_2)$$

Consequently, if each cord had a spring of yield-constant Y_1 or Y_2, and a friction-constant of f_1 or f_2, besides being coupled to the pulley, we should have for the total forces F_1 and F_2 required in each circuit—

$$\begin{aligned}\mathsf{F}_1 &= \frac{1}{Y_1}s_1 + f_1v_1 + F_1\\ &= \frac{1}{Y_1}s_1 + f_1v_1 + \frac{m}{4}a_1 + \frac{m}{4}a_2\end{aligned}$$

and similarly $$\mathsf{F}_2 = \frac{1}{Y_2}s_2 + f_2v_2 + \frac{m}{4}a_2 + \frac{m}{4}a_1$$

If we compare these equations with the electrical ones on p. 206 we see that they are exactly similar in form if we consider L_1, L_2, and M as each equal and corresponding to $\dfrac{m}{4}$. The model therefore corresponds exactly to the electrical case of two coils wound on a closed iron circuit, having no magnetic leakage and equal numbers of turns.

The phenomena of leakage can be exactly imitated by removing part of the weight from the car and adding weights to the cords direct, while the effect of unequal numbers of turns in the primary and secondary can be reproduced by using two pulleys of different diameters turning together round the same axis.* Fig. 131 shows an arrangement which corresponds very closely to an ordinary transformer in which there is a certain amount of primary and secondary leakage, which produces what is called the primary and secondary leakage inductance, and in which there is a transforming down of force or voltage in the ratio of the diameters of the pulleys, and a transforming up of the motion or current in the same ratio. This model corresponds exactly to Fig. 132, which is the same as the leaky transformer shown in Fig. 128, except that here, instead of showing the leakage lines in their true place as in Fig. 128,

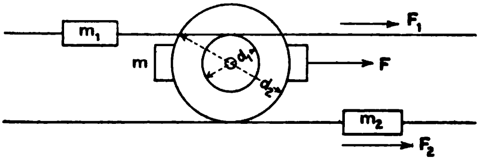

FIG. 131.—Mechanical model of step-down transformer.

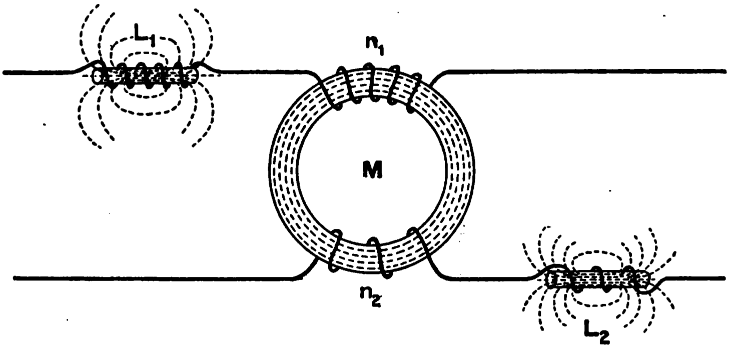

FIG. 132.—Step-down transformer with magnetic leakage shown separately.

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* Hysteresis in the core can be simulated by ordinary solid friction in the pulley axle, and eddy currents by a second circuit in parallel with cord 2.

they are represented as if they were separated from the core. A little thought will show that this is equivalent to the leaky transformer, and that therefore magnetic leakage is equivalent to separate inductances in the primary and secondary circuits.

Another model, based on the original mechanical analogies of Clerk Maxwell, is shown in Fig. 133, and is obtainable from German instrument makers. It consists of two shafts fixed together in the form of a cross, the shorter portion of which turns in the bearings B_1B_2, and the longer of which can have weights W W of any size fixed on it. Four bevel wheels, each running loosely on their respective shafts, gear into one another like the differential gear of an Aron meter or of a tricycle. The cords representing the primary and secondary

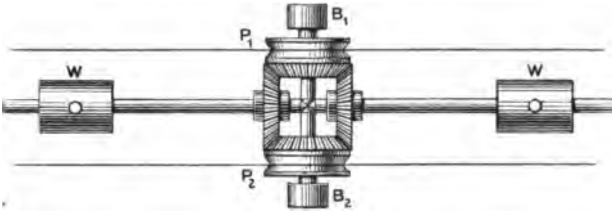

FIG. 133.—Model of mutually inductive circuits.

circuits can run round pulleys P_1P_2 attached to two of the bevel wheels as shown. Both of these cords wrap round the pulleys in the same manner, not in opposite directions, as in Fig. 130. If cord 1 is moving uniformly and turning pulley P_1 round steadily, while pulley P_2 is stationary, it simply revolves the other two bevel wheels, and the long shaft and weights turn at half the rate of pulley P_1. Since, however, there is no acceleration, the inertia of these weights will have no effect, and if the mechanism has no friction no force will be imparted to pulley P_2 or the secondary cord. If, however, cord 1 and pulley P_1 accelerates, the weights W will resist acceleration and a force must be communicated from P_1 to the bevel wheels on the long shaft, and these wheels will in their turn communicate the same force in the opposite direction to pulley P_2. Hence if cord 1 accelerates, cord 2 receives a backward force or E.M.F. which will cause it to move backwards if free to do so while the primary acceleration lasts. On the other hand, if cord 1 has been running steadily and then slows down, the weights W tend to run on and thus communicate a forward force to both pulleys, and if cord 2 is free to move it will start forwards like the secondary current at break. The theory of this mechanism would be found to be of the same nature as the other, but we have preferred to deal principally with the truck model as harmonising more perfectly with our former models, and as only requiring reasoning about straight line instead of rotatory motion.

Alternate-Current Circuits with Mutual Induction.—We have hitherto been discussing the effects of mutual induction as regards the starting and stopping of the primary or secondary current. It remains to consider what happens when a simple harmonic alternating current is passed through the primary.

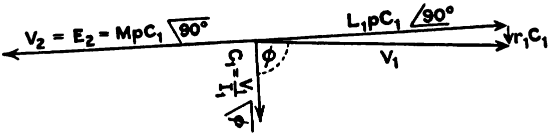
$V_2 = E_2 = MpC_1 \overline{\diagdown} 90^\circ$
$L_1pC_1 \angle 90^\circ$
r_1C_1
V_1
ϕ
$C_1 = \frac{V_1}{I} \overline{\diagdown} \phi$

FIG. 134.—Transformer on open circuit.

The E.M.F. induced in the secondary due to mutual induction was $E_2 = - MA_1$, and similarly the E.M.F. in the primary due to the secondary current was $E_1 = - MA_2$. Now if we have a primary current of effective value $\hat{C}_1$ and of frequency n, we know that $\hat{A}_1 = p\hat{C}_1 \angle 90^\circ$ or $= jp\hat{C}_1$, and hence—

$$\hat{E}_1 = - Mp\hat{C}_2 \angle 90^\circ = Mp\hat{C}_2 \overline{\diagdown} 90^\circ \text{ or } -jMp\hat{C}_2 . \quad (48)$$

$$\text{and } \hat{E}_2 = Mp\hat{C}_1 \overline{\diagdown} 90^\circ = -jMp\hat{C}_1 \quad . \ . \ . \ . \ . \ . \ . \quad (49)$$

Let us now take the case of a simple transformer consisting of primary and secondary coils of n_1 and n_2 turns respectively, as in Fig. 127, and apply a voltage V_1 to the primary. Then if L_1 is the primary inductance in henries and r_1 its resistance in ohms, there will be a primary current of $\hat{C}_1 = \frac{\hat{V}}{I} \overline{\diagdown} \phi$ where $I = \sqrt{r_1^2 + L_1^2p^2}$ and $\tan \phi = \frac{L_1p}{r_1}$. When the coils are wound on an iron core, as in this case, L_1p is generally large in comparison with r_1, and the current therefore lags by nearly 90°, as shown in Fig. 134. The secondary E.M.F. is then, as we have just seen, $\hat{E}_2 = Mp\hat{C}_1 \overline{\diagdown} 90^\circ = \frac{Mp}{I}\hat{V}_1 \overline{\diagdown} (\phi + 90^\circ)$, and is therefore represented by a vector at right angles to $\hat{C}_1$, or lagging $\phi + 90^\circ$ behind $\hat{V}_1$. If the primary resistance r_1 is negligible, $I = L_1 p$ and ϕ becomes 90°, hence—

$$\hat{E}_2 = \frac{Mp}{L_1 p}\hat{V}_1 \, \Gamma \, (90^\circ + 90^\circ) = -\frac{M}{L_1}\hat{V}_1$$

or the secondary effective E.M.F. is opposite in phase to V_1 and $\frac{M}{L_1}$ times its value. But on referring to equations (42) and (43), p. 204, we saw that

$$M = \frac{4\pi}{10^9 \mathsf{R}} n_1 n_2 \quad \text{and} \quad L_1 = \frac{4\pi}{10^9 \mathsf{R}} n_1^2$$

hence $$\frac{M}{L_1} = \frac{n_2}{n_1} \quad \text{and} \quad \hat{E}_2 = -\frac{n_2}{n_1}\hat{V}_1 \quad \text{or} \quad \frac{\hat{E}_2}{\hat{V}_1} = -\frac{n_2}{n_1}$$

the ratio of the effective values of the secondary and primary voltages, when no current is taken from the secondary, is therefore equal to $\frac{n_2}{n_1}$ or the ratio of secondary to the primary turns, which is called the ratio of transformation of the transformer. This result is perfectly natural, as since there is the same alternating magnetic flux in the iron core cutting the two coils, the voltages induced in them must be proportional to the numbers of their turns.

If we take our cord model in Fig. 131, clamp the second cord so as to prevent its having any motion or current, and oscillate the first cord backwards and forwards, it will be readily seen that there will be forces in the second cord exactly opposite to those in the first, and that $\frac{F_2}{F_1} = \frac{d_1}{d_2}$, the ratios of the diameters of the pulleys. Hence the ratio of the diameters of the pulleys corresponds to the ratio of transformation of the transformer, except that it must be taken the inverse way, as it is $\frac{d_1}{d_2}$ instead of $\frac{n_2}{n_1}$.

Transformer under Load.—Now suppose that we load the transformer by drawing current from the secondary. There are two ways of showing what will happen in this case, and we shall first use the one which directly follows the principles we have been studying. The most convenient way of attacking the problem is to suppose that the transformer is giving a certain secondary voltage represented by the vector $\hat{V}_2$ in Fig. 135, and a certain current $\hat{C}_2$, which we have here represented as lagging behind $\hat{V}_2$, and then to find out what must be the primary voltage and current in order to produce them. Now the E.M.F. $\hat{E}_2$ induced in the secondary by the primary current has to do three things: (*a*) supply the secondary P.D. V_2, (*b*) force the current through the secondary resistance r_2, and (*c*) force the current through the secondary reactance L_2p. We therefore put on the end of the secondary voltage vector $\hat{V}_2$ a vector $r_2\hat{C}_2$ parallel to or in phase with $\hat{C}_2$ representing the secondary resistance drop; and on the end of this again the secondary reactance drop $L_2p\hat{C}_2$ leading in front of $\hat{C}_2$ by 90°. The vector sum of these three voltages must be the secondary E.M.F. $\hat{E}_2$ induced by the primary current. But this vector must be $Mp\hat{C}_1$ and must lag 90° behind the primary current, as shown on p. 214; hence $\hat{C}_1 = \frac{\hat{E}_2}{Mp}$ and leads 90° in front of $\hat{E}_2$. We can therefore set off a vector $\hat{C}_1$ perpendicular to $\hat{E}_2$ as the primary current.

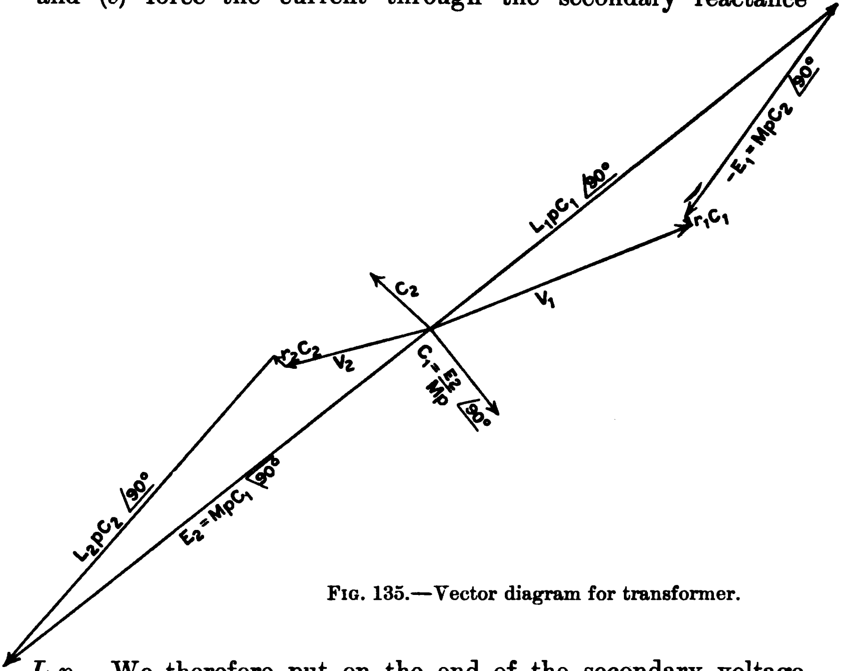

FIG. 135.—Vector diagram for transformer.

The primary voltage $\hat{V}_1$ has to force the current through the primary coil against the secondary induced E.M.F., and the primary resistance and reactance drops, or it is the vector sum of $Mp\hat{C}_2 \angle 90^\circ$ (leading 90° instead of lagging 90°, because we are dealing with the voltage required to *overcome* the voltage induced by the secondary current), $r_1\hat{C}_1$, and $L_1p\hat{C}_1 \angle 90^\circ$.

It will be seen that according to this diagram the primary P.D.'s and currents are approximately opposite in phase to one another, just as we saw that $\hat{V}_2$ and $\hat{V}_1$ were opposite in phase in Fig. 134.

Total and Leakage Inductance.—We have already drawn attention more than once to the fact that the inductance of the primary or secondary coils of a transformer is made up of two parts, the first due to the flux which passes through both coils, and the second due to the flux which passes through one coil only and which we termed the leakage flux. The total inductance of each coil is the total number of lines linked with it per ampere due to the total flux, both that which passes through both coils and the leakage flux belonging to the coil considered; while the leakage inductance of each coil is the number of lines of force per ampere due to the leakage flux which cuts that coil only. This latter flux could be represented as if there were another separate coil of the same number of turns, with the leakage flux passing through it, in series with the transformer coil as in Fig. 132.

Now if we consider the equations on p. 206, and the vector diagram for the transformer which we have just constructed, we shall see that *we have been treating the voltages just as if the two transformer coils had mutual induction without any self-induction, and as if each coil had an inductance equal to the total inductance in series with it.* In fact, we have supposed that each circuit had a very large E.M.F. induced in it by mutual induction, and that a considerable part of this E.M.F. was used up or neutralised by the self-induction of the coil.

There is nothing wrong about the results obtained from these equations or diagrams. But if we look into what is really happening a little more closely we shall see that we have not got quite a true picture of these voltages. The reason is that both self and mutual induction imply magnetising the core, and the core is not simply magnetised to a high degree by the primary current, thus inducing a large secondary E.M.F., and at the same time acting as the carrier of the flux due to the self-induction of the secondary coil. The two coils with the primary and secondary currents act together on the core producing *one flux only*, and the only separate self-induction fluxes are those due to the leakage lines.

The distinction can be readily seen by considering a dynamo having an E.M.F. of 150 volts, charging cells with a back E.M.F. of 100 volts through a total resistance of ten ohms. We could find the current quite correctly in the following manner:

$$\text{Current due to dynamo}\quad C_1 = \frac{E_1}{r} = \frac{150}{10} = 15 \text{ amperes}$$

$$\text{Back current due to cells}\quad C_2 = \frac{E_2}{r} = \frac{100}{10} = 10 \text{ amperes}$$

$$\text{Resultant current} = C_1 - C_2 = 15 - 10 = 5 \text{ amperes}$$

In other words, we have considered the actual current in the circuit as being made up of two parts, one due to the dynamo as if it were alone present, and the other in the opposite direction due to the cells as if they alone existed. There is nothing wrong about this method of solving the problem so far as the result is concerned, but it is much simpler, and more in accordance with the facts, if we say that we have a resultant E.M.F. $E_1 - E_2 = 150 - 100 = 50$ volts, and that this produces an actual current of $\frac{50}{10}$ or 5 amperes, instead of a resultant current of 5 amperes due to a forward current of 15 amperes and a back current of 10 amperes.

So it is with our coils. There is nothing wrong philosophically in imagining two E.M.F.'s due to a forward and a back flux due to mutual and self-induction (except when we come to consider the saturation of an iron core), but the fact is that the magnetising forces of the primary and secondary coils unite in producing a single flux which induces a smaller E.M.F. in the primary and secondary coils. If we separate the primary inductance L_1 into two portions L_1' and l_1, the first of which passes through both circuits and the second is the leakage inductance, the E.M.F. induced in the primary is not $\hat{E}_1 = M\hat{C}_2 \bar{\vee}\, 90°$, but

$$\hat{E}_1' = M\hat{C}_2 \bar{\vee}\, 90° + L_1'\hat{C}_1 \bar{\vee}\, 90°$$

and the only truly self-inductive effect is that of the leakage inductance, which really acts as if it were separate. In Fig. 136 the diagram has been modified in accordance with these principles. Starting with $\hat{V}_2$ and $\hat{C}_2$ as before, we add $r_2\hat{C}_2$ and $l_2p\hat{C}_2$ on to $\hat{V}_2$, giving as the real secondary E.M.F. $\hat{E}_2'$, which lags 90° behind the resultant core flux. This core flux requires a certain magnetising force to produce it, which is in phase with the flux, and is therefore represented by a current $\hat{C}_m$ known as the magnetising current, perpendicular to $\hat{E}_2'$. Since the resultant flux cuts both the primary and secondary coils, the E.M.F. induced in each will be the same if the number of turns are equal, but as the primary E.M.F. has to be overcome by the primary voltage we set it out equal and opposite to $\hat{E}_2$. The primary and secondary currents together have to produce a magnetising effect equal to the magnetising current $\hat{C}_m$. Hence, if we draw a line equal and opposite to $\hat{C}_2$ and combine it with $\hat{C}_m$, we get the primary current $\hat{C}_1$. Adding $r_1\hat{C}_1$ and $l_1p\hat{C}_1$ ∠ 90° we get the primary voltage $\hat{V}_1$, which corresponds with that obtained in Fig. 135. The dotted lines in Fig. 136 show the portions of Fig. 135 which have been done away with by considering the combined instead of the component fluxes.

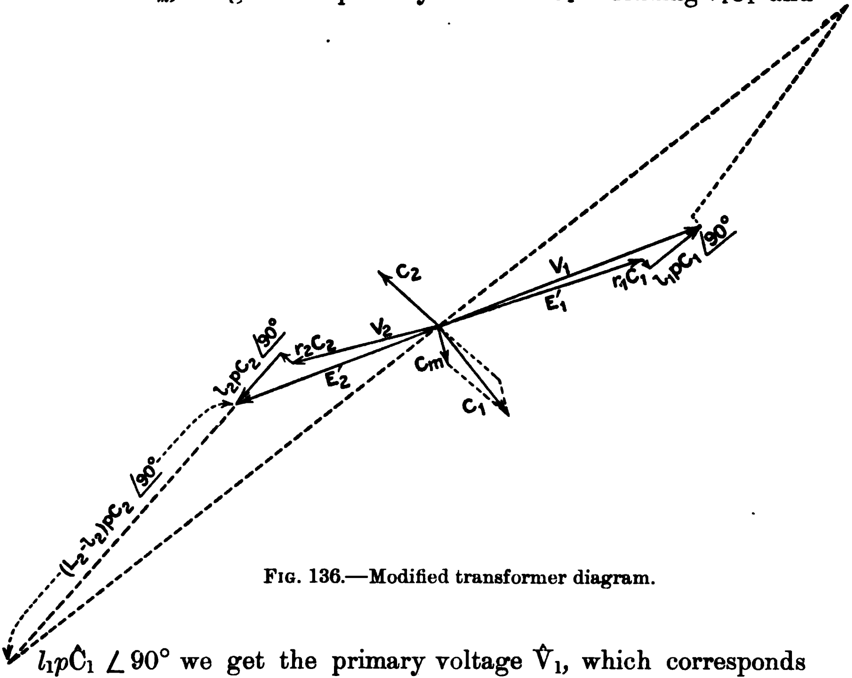

FIG. 136.—Modified transformer diagram.

If the magnetising current C_m is kept constant E_1' and E_2' are constant, and we can see how the primary current and the voltages vary, as the secondary current is altered. We can also modify the diagram slightly to allow for hysteresis and eddy currents in the core, &c., but this must be reserved for a future volume.

The student is strongly advised to thoroughly master the distinction between total and leakage inductance, as it is of the highest importance in practice. In the study of the regulation of alternators, motors, and transformers, the most serious mistakes have been made from want of appreciation of this matter

APPENDIX TO CHAPTER XIV

THEORY OF CORD MODEL OF MUTUALLY INDUCTIVE CIRCUITS FOR UNEVEN RATIO

LET d_1 and d_2 be the diameters of the pulleys; F_1 and F_2 the forces imparted to the pulleys; m the mass of the car; F the force on the car, as in Fig. 131. Then by ordinary mechanical principles—

$$F_1 + F_2 = F \quad \text{and} \quad F_1 d_1 - F_2 d_2 = 0 \quad \text{or} \quad F_1 d_1 = F_2 d_2$$

If s_1 and s_2 are the displacements of the two cords, s that of the car, and θ the angle turned by the pulley—

$$s_1 = s + \frac{d_1}{2}\theta \qquad\qquad s_2 = s - \frac{d_2}{2}\theta$$

Hence $$d_2 s_1 + d_1 s_2 = (d_1 + d_2)s$$

or $$s = \frac{d_2 s_1 + d_1 s_2}{d_1 + d_2}$$

from which $$a = \frac{d_2 a_1 + d_1 a_2}{d_1 + d_2}$$

From the first two equations we have—

$$F_1 + \frac{d_1}{d_2} F_1 = F = ma$$

hence $$F_1 = \frac{d_2}{d_1 + d_2} ma$$

and putting in the value obtained above for a—

$$F_1 = \frac{d_1}{d_1 + d_2} m \frac{d_2 a_1 + d_1 a_2}{d_1 + d_2}$$

or $$F_1 = m\frac{d_2^2}{(d_1 + d_2)^2} a_1 + m\frac{d_1 d_2}{(d_1 + d_2)^2} a_2$$

and similarly $$F_2 = m\frac{d_1^2}{(d_1 + d_2)^2} a_2 + m\frac{d_1 d_2}{(d_1 + d_2)^2} a_1$$

If the first circuit is taken as the primary, the force required to produce the given motion of the car will be F_1, and consists of two parts—the first, $m\frac{d_2^2}{(d_1+d_2)^2}a_1$, which is required to accelerate the first cord, and therefore corresponds to a mass or inductance of the first circuit $L_1 = m\frac{d_2^2}{(d_1+d_2)^2}$; and a second, $m\frac{d_1d_2}{(d_1+d_2)^2}a_2$, to overcome the back force due to the acceleration of the second cord, and hence corresponds to a mutual inductance M of $m\frac{d_1d_2}{(d_1+d_2)^2}$. In like manner the second cord moves as if it had a mass or self-inductance of $L_2 = m\frac{d_1^2}{(d_1+d_2)^2}$ and a mutual inductance of $M = m\frac{d_1d_2}{(d_1+d_2)^2}$. We therefore see that the mutual inductance between the primary and secondary is the same as that between secondary and primary, as we found for the electrical circuits, and further that—

$$M^2 = m^2\frac{d_1^2d_2^2}{(d_1+d_2)^4} = L_1L_2 \quad \text{or} \quad m\frac{d_2^2}{(d_1+d_2)^2} \times m\frac{d_1^2}{(d_1+d_2)^2}$$

corresponding to what we found for the mutually inductive circuits without magnetic leakage on p. 205. Also $\frac{L_1}{L_2} = \frac{d_2^2}{d_1^2}$, corresponding to the relation $\frac{L_1}{L_2} = \frac{n_1^2}{n_2^2}$ for the electrical case.

If there is magnetic leakage, this corresponds, as we saw on p. 212, to external masses or to inductances l_1 and l_2 in the separate cords, and if there is also resistance or friction f_1 and f_2 in the two circuits, we shall have—

$$F_1 = f_1v_1 + l_1a_1 + L_1a_1 + Ma_2$$

or

$$= f_1v_1 + (L_1 + l_1)a_1 + Ma_2$$

and similarly

$$F_2 = f_2v_2 + (L_2 + l_2)a_2 + Ma_1$$

By arranging for the central mass to move with friction, an almost perfect imitation of an actual transformer can be constructed, which will illustrate the 'no load' and 'short circuit' tests which are made on commercial transformers, exactly.

CHAPTER XV

ELECTRICAL OSCILLATIONS

HAVING now covered the groundwork of alternate-current theory, we come to a subject of great interest and importance. In Chapter V, in dealing with combinations of friction, inertia, and elasticity, we observed that oscillations were produced by suddenly loading a spring, or by suddenly charging or discharging a condenser; and again in Chapters VI and X, it followed from the principles of simple harmonic motion and of resonance that a circuit containing inductance and capacity is like a musical instrument tuned to a certain note which it emits, or to which it responds by getting into vibration whenever that note is sounded in the vicinity.

As we saw in Chapter VI, it is an extremely simple thing to find out the kind of oscillation which takes place when a spring is stretched or a condenser is discharged, provided that there is no friction or resistance. For if we take a spring with a yield-constant Y coupled to a mass of m engineers' units, stretch it and let it go, we shall have $F = F_1 + F_3 = \frac{1}{Y}s + ma = 0$, since when released there is no external force acting. But this means that $ma = -\frac{1}{Y}s$ or $a = -\frac{1}{Ym}s$.

Now in Chapter VI we investigated the relations between the displacement, velocity, and acceleration in the case of simple harmonic motion and obtained the result $a = -p^2s$, or that the acceleration was proportional to the displacement at every instant, and opposite to it in sign or direction. The quantity $p = 2\pi n$ where n is the number of complete periods of the motion per second. When this is compared with the above equation $a = -\frac{1}{Ym}s$, it is seen that the two equations would be exactly alike if $\frac{1}{Ym} = p^2$.

Although we have shown that for every simple harmonic motion the acceleration is proportional to and opposite to the displacement, we have not proved mathematically that every motion in which this relation between the acceleration and displacement is found is a simple harmonic motion. But a little reflection will show that there can be only one kind of motion in which this relation is found; for if the displacement, velocity, and acceleration are known at any instant, there is only one way in which the displacement of the body can alter.

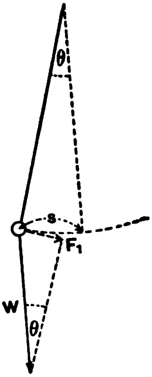

FIG. 137.—Action of pendulum.

Taking it for granted, therefore, that whenever a body moves so that its acceleration is proportional and opposite to its displacement, its motion is simple harmonic, we know that the motion of the spring and weight must be simple harmonic, and that $p_2 = \frac{1}{Ym}$ or $p = \frac{1}{\sqrt{Ym}}$ and $n = \frac{1}{2\pi\sqrt{Ym}}$. If $\mathcal{T}$ is the time for a complete oscillation and there are n oscillations in a second, it is evident that $\mathcal{T} = \frac{1}{n} = 2\pi\sqrt{Ym}$, or the periodic time of oscillation of a spring and weight is 2π times the geometric mean of its yield-constant and mass. In the case of an ordinary pendulum (Fig. 137) the spring force is provided by the earth's attraction on the bob, which tries to bring it back to the centre. The downward force on the bob is $W = mg$, if W is the weight or force in pounds, m the mass in engineers' units, and g the attractive force of the earth for one engineers' unit, = 32·2 lbs. in Great Britain. But this downward force can be separated into two parts, one $W \cos\theta$ in the direction of the cord, and the other $F_1 = W \sin\theta$ at right angles to it or in the direction of its motion. The force along the cord does not affect the motion, as it is at right angles to the path, and if the pendulum only moves over a small arc, θ is small, and $\sin\theta = \theta = \frac{s}{l}$, where s is the displacement and l the length of the pendulum. Hence $F = W\sin\theta = \frac{mg}{l}s$, and since, when the pendulum swings to one side, the force is towards the centre or in the opposite direction to the displacement, $F = -\frac{mg}{l}s$. The pull of the earth on the bob of a pendulum is therefore proportional to the displacement, and acts like a spring of yield-constant $\frac{1}{Y} = \frac{mg}{l}$ or $Y = \frac{l}{mg}$. Consequently the bob of the pendulum should move with a simple harmonic motion for which $\mathcal{T} = 2\pi\sqrt{Ym} = 2\pi\sqrt{\frac{l}{mg} \times m} = 2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum in feet, and $g = 32{\cdot}2$ in London. This formula gives us the exact time of swing of a pendulum for small angles of swing, and can be readily checked by anyone.

In the case of the spring and weight it will also be seen that the oscillations are more rapid, the smaller either the yield-constant or the mass. If a spring which yields a foot for a force of 1 lb., or for which $Y = 1$, is loaded with an engineers' unit, we should have $\mathcal{T} = 2\pi\sqrt{1 \times 1} = 6{\cdot}28$ secs., or $n = \frac{1}{6{\cdot}28} = {\cdot}16$ period per second. But if the spring only yields ·01 ft. per lb., we have $n = \frac{1}{2\pi\sqrt{{\cdot}01 \times 1}} = 1{\cdot}6$ periods per second. Reducing the mass to 1 lb. or ·03 engineers' unit, $n = \frac{1}{2\pi\sqrt{{\cdot}01 \times {\cdot}03}} = 9{\cdot}2$. Finally, if we take a bar which yields about a thousandth of an inch or, say, a ten-thousandth of a foot for a pound, and having a mass of 1 lb., we should have $n = \frac{1}{2\pi\sqrt{{\cdot}00010 \times {\cdot}03}} = 92$, which would give a fairly high musical note. As a matter of fact, the note would be higher than this owing to the mass being distributed all along the rod, instead of being concentrated at its end.

Next turning to the electrical case of a condenser discharging through an inductance, $V = V_1 + V_3 = \frac{1}{K}Q + LA = 0$, from which $A = -\frac{1}{KL}Q$, and we have a simple harmonic motion for which $p^2 = \frac{1}{KL}$ and the frequency $n = \frac{1}{2\pi\sqrt{KL}}$. The relation $p^2 = \frac{1}{KL}$ is the same as $Lp = \frac{1}{Kp}$ which we obtained for a circuit in resonance (see Chap. XI), and shows that resonance occurs when the supply frequency corresponds to the natural oscillation frequency of the condenser and inductance, as previously shown.

Now let us see what the oscillation frequencies will be in various cases of condensers and inductances. Since $p^2 = \frac{1}{KL}$ where K is in farads and L is in henries, $p^2 = \frac{10^6}{KL}$ when K is in microfarads, $\therefore p = \frac{1000}{\sqrt{KL}}$ and $n = \frac{1000}{2\pi\sqrt{KL}} = \frac{159{\cdot}2}{\sqrt{KL}}$. Hence a 1-mfd. condenser with a coil of 1 henry inductance would have an oscillation frequency of 159·2 ~ per sec. A 10-mfd. condenser with the same coil would have an oscillation frequency of $\frac{159{\cdot}2}{\sqrt{10}} = 50{\cdot}4$ ~ per sec., and would consequently be practically in resonance with a 50 ~ circuit.

Next suppose that we decrease the capacity and inductance considerably. An ordinary pint size Leyden jar has a capacity of about ·0018 mfd. Suppose this is connected to a coil having an inductance of 10 millihenries or ·01 henry. The frequency $n = \frac{159{\cdot}2}{\sqrt{{\cdot}01 \times {\cdot}0018}} = 37{,}500$ ~ per sec., which is extremely high in comparison with ordinary supply frequencies. But let us go further and use the same jar with a single loop of wire about

1 ft. in diameter and $\frac{1}{10}''$ thick. Such a loop would have an inductance of something like ·6 microhenry or $\frac{\cdot 6}{10^6}$ henry, and the vibration frequency has now risen to $\frac{159{,}200}{\sqrt{\cdot 6 \times \cdot 0018}} = 4{\cdot}85$ *millions* per sec. Even this is not our limit. Instead of the Leyden jar we can use a couple of plates, say 20 cm. diameter and 1 cm. apart, as our condenser, which would have a capacity of about 3×10^{-5} mfds., and this with the same loop would have an oscillation frequency of $\frac{159{,}200}{\sqrt{\cdot 6 \times 3 \times 10^{-5}}} = 37{\cdot}5$ millions. By separating our plates still further, the capacity can be reduced until the oscillation frequency runs into hundreds of millions per sec. Fig. 138 shows the dumb-bell oscillator of Hertz, consisting of two spheres 30 cm. or about 1 ft. diameter, and having two rods each 3 mm. diameter and 75 long connecting

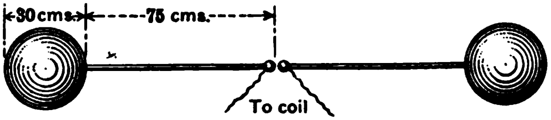

FIG. 138.—Hertz oscillator.

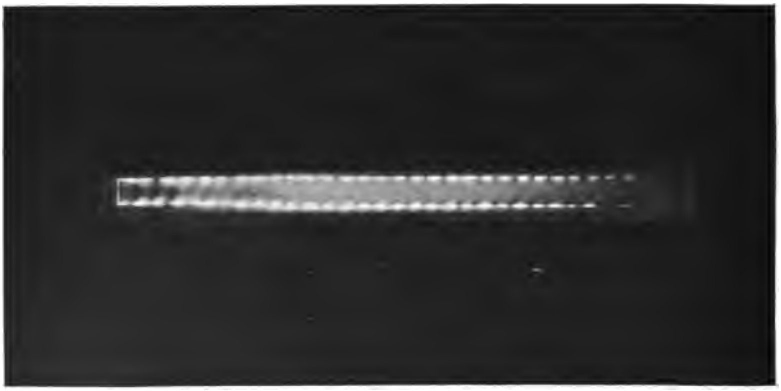

FIG. 139.—Photograph of oscillatory spark in revolving mirror.
$K = \cdot 0072$; $L = \cdot 022$; $n = 12{,}600 \sim$ per sec.

them, with a spark-gap in the middle. The oscillation frequency of this arrangement is almost exactly 100,000,000 per second.

The oscillatory character of the discharge from condensers and Leyden jars was first shown mathematically by Lord Kelvin, and verified experimentally by Feddersen, who observed the spark in a rapidly revolving mirror, and saw that there was really a large number of sparks. Fig. 139 shows a photograph of a spark taken in this manner.

The introduction of wireless telegraphy by Marconi in 1896, and the rapid extension of both wireless telegraphy and telephony, has made the production and study of rapid electrical oscillations a matter of great practical importance, and we shall now consider the various ways by which these oscillations may be stimulated and maintained. For this purpose we shall find it useful to first consider the stimulation of mechanical oscillations, and this can be most clearly explained with reference to ordinary musical instruments.

Impulsive Stimulus.—The most common way of obtaining mechanical oscillations is by imparting a sudden impulse or blow to a combination of elasticity and inertia, as in a tuning fork or bell or a stretched piano string, or by suddenly releasing a string as in the harp or the *pizzicato* playing of the violin. The result of this impulse is to set the body or string in oscillation and to leave it free to vibrate. The vibrations so produced, however, die away more or less rapidly partly because of internal or other friction, but partly also because they communicate sound vibrations to the air around or radiate sound energy in all directions. This energy must be taken from their energy of motion, and therefore this motion must become less. If a more or less regular supply of these oscillations is required, the impulses must be repeated fairly frequently, giving rise to successive sets or trains of damped oscillations, as in the mandoline (see Fig. 140).

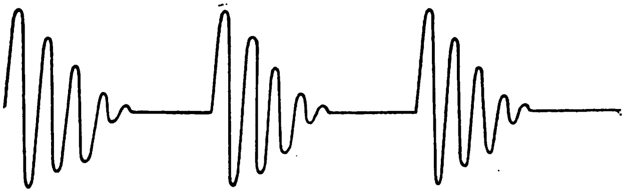

FIG. 140.—Successive trains of damped oscillations.

Electrical oscillations are similarly most commonly excited by impulsive methods, and especially those corresponding to the *pizzicato* method, in which there is a slowly applied force with sudden release. If an ordinary Leyden jar is connected to a spiral of wire as in Fig. 141, and the jar is slowly charged from an electrical machine, a time arrives when the voltage becomes high enough to break down the air-gap, and the charge is suddenly released and oscillates like the released string. Fig. 142 shows a very common arrangement in which two Leyden jars are connected in series with one another, and with a spiral, to an ordinary induction coil. At each break of the primary current the jars are charged more or less rapidly, until the P.D. is sufficient to break across the spark-gap formed by the rods and balls, nearly connecting the knobs of the Leyden jars together. The oscillations produced by the discharging of the jars then pass round this circuit but not through the induction coil, as its

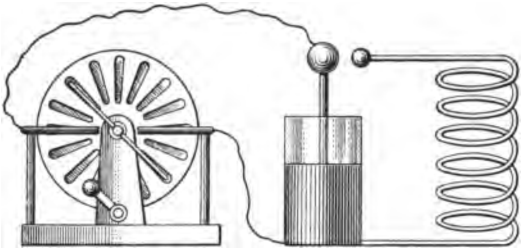

FIG. 141.—Production of electrical oscillation.

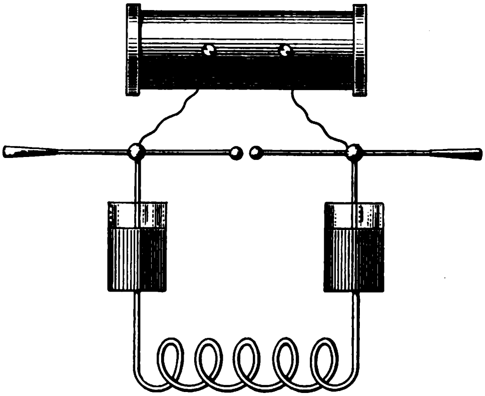

FIG. 142.—Electrical oscillator.

inductance is too high to allow any appreciable current to pass at this high frequency.

Continuous Oscillations.—There are two classes of musical instruments which produce a continuous sound: the stringed instruments which are played with a bow, such as the violin and 'cello, and the wind instruments. The action of the former is more easy to understand. We have here the stretched string with its elasticity and inertia, and a bow which is drawn across the string and sets it in vibration.

What is the action which takes place between the bow and the string? There is evidently friction between them, but is that competent to produce the vibration? In Chapter I we considered the various kinds of friction, solid and fluid, and we came to the conclusion that there were, roughly speaking, three laws of friction: solid friction in which the force was independent of the speed, fluid friction at low speeds in which the force was proportional to the speed, and fluid friction at high speeds when the force was proportional to the square of the speed. It is not difficult to see that if any of these laws are true, friction will not keep the string in continuous oscillation. If the solid-friction law is true, the drawing of the bow across the string is equivalent to applying a steady force tending to pull or push the string to one side, and the result will be that it may oscillate at first, but it will soon settle down to its new position, and the sound will cease. We know that the drawing of, say, a metal rod across a violin string will not make it give a musical sound.

If either of the other two laws of friction are true, the tendency to oscillation will be even less. For if we suppose that the string is at first at rest, and the bow is drawn steadily across it, there will be a certain frictional force which will tend to move the string in the direction of the motion of the bow as before. But as soon as the string starts to follow the bow, the speed of the bow relatively to that of the string gets less, and hence if the force is proportional to the velocity, or to the square of the velocity, it becomes less and less as the speed of the string increases. There will therefore be less tendency to cause the string to overshoot its mark and to vibrate, and even if it does so, and the elastic force of the string begins to pull it back and to slow down its motion, then the relative velocity gets greater and the force on the string again increases, thus tending to neutralise the recoil.

In order to keep up the motion of a pendulum or of a spring and weight, the best way is to give a force or impulse at the beginning of its stroke or when its velocity is highest, *i.e.* when its velocity relative to the bow is least. Fluid friction has just the opposite effect to this, and consequently if a violin bow is covered with oil, however viscous the oil may be, and consequently however great the friction, the string will not vibrate.

It will be evident from the above that, in order to make the spring vibrate continuously by the friction of the bow, we must have the greatest force when the relative velocity of the string and bow is least, or that the force of friction should be *inversely* proportional to some power of the velocity. Now, as a matter of fact, solid friction between roughened surfaces behaves somewhat in this way. If we take a block of wood resting on a board and pull it along, we shall find that it wants a decidedly greater force to start it than to keep it in motion, quite apart .from the effects of inertia. The reason is that when the block is at rest a number of its irregularities fit into irregularities on the surface of the plank and more or less lock it in position. But when the force is increased sufficiently to break through or flatten the irregularities, or to make the block surmount them, it will move forwards, and its inertia will prevent it from dropping into the other irregularities as it rapidly passes over them. It will be seen that the amount of this locking will be less and less as the speed gets greater, or that the force of friction will diminish as the speed increases, as in the curve marked 'Adhesion' in Fig. 3. The more irregular the surfaces, as long as the irregularities are very numerous and uniformly distributed over the surfaces, the greater will be the effect.

Now, how do we actually set the violin string in vibration ? We use a bow made of horsehair and rub it on a block of resin before playing. If a hair of a well-resined bow is examined under the microscope (Fig. 143) it will be seen that it is covered with little flakes or jagged bits of resin, which give it a saw-like appearance. The irregularities so produced catch the string and draw it forwards as long as the elastic force is not too great, but as soon as this passes a certain amount the string flies back, and, as its motion during the back swing is opposite to that of the bow, the relative velocity is great and the friction consequently small. On again reversing its motion the friction increases and the bow imparts a new impulse to the string, and so on. The result is that while the bow is drawn steadily in one direction a series of impulses are given to the string in somewhat the same way as a pendulum can be kept swinging by giving it a blow with the hand every time it passes the centre in the same direction.

It will at once be noticed that the kind of vibrations produced by the impulsive or *pizzicato* and continuous or *arco* (bow) methods will differ. In the impulsive method a single impulse is given and the string is left free to vibrate. The result is a set or train of free but damped vibrations almost exactly simple harmonic in form, but dying away before the next impulse is given. On the other hand, the bow gives an impulse at every period, and there is therefore no dying away, but at the same time the string is never left to vibrate freely, and is always more or less constrained by contact with the bow. The vibrations are then said to be 'forced' in contradistinction to 'free' vibrations, and we should expect them to be more or less distorted from the simple harmonic form. It is this distortion which causes that peculiar wailing quality in the note of the violin, and distinguishes it at once from the same note sounded on a piano, in which the vibrations are free.

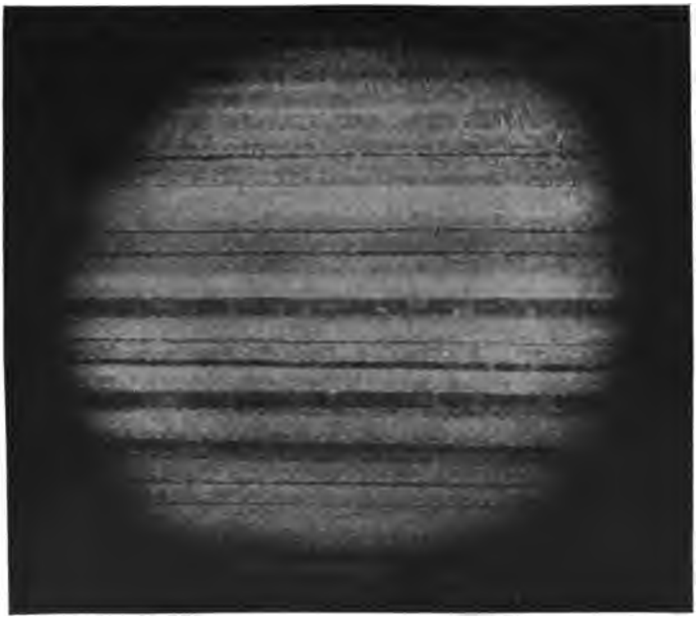

Fig. 143.—Hair of violin bow under microscope.

It is thus seen that the production of continuous mechanical oscillations depends upon applying a force to a combination of elasticity and inertia through the agency of a friction in which the force varies inversely with the velocity. Our analogies show us that we must seek to produce continuous electrical oscillations in the corresponding manner, *i.e.* by applying an electrical force to a combination of capacity and inductance through the agency of an electrical friction in which the electrical force or P.D. is inversely as the electrical speed or current.

Now, is there any example of an electrical resistance which diminishes with the current? We had one example in Chapter I in the case of a carbon filament glow-lamp in which the resistance gets less as the temperature gets higher with increase of current. But this only means that the P.D. increases more slowly than the current, not that it actually falls as the current is increased.

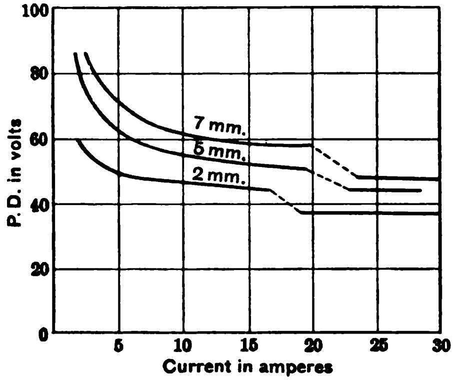

FIG. 144.—Arc characteristics.

There is one case, however, which corresponds exactly to what we are seeking, and that is a simple air-gap. Such a gap, a centimetre long, is practically a perfect insulator for pressures of a few hundred volts, and requires a P.D. of about 30,000 volts to overcome its stationary friction. But when this is done and a current is passed through it, the P.D. between the electrodes rapidly falls as the current increases, as in the curve marked 'Arc' in Fig. 5, until with about 10 amperes the P.D. is only about 60 volts.* Fig. 144 gives some characteristic curves, taken from ordinary electric arcs with solid carbons by Mrs. Ayrton, which clearly show the drop of P.D. as the current increases. If the positive electrode is made of a block of copper which is always kept cool, and the negative electrode is of carbon, while the arc is formed in a space filled with hydrogen or coal gas, as employed by Poulsen, the curve is even steeper, as shown in Fig. 145. It is this negatively sloping characteristic of the arc that makes it so unstable and renders it necessary to have a steadying resistance and a sensitive mechanism to control it.

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* The reason for this is akin to that we gave above for the drop of the force of friction with increased velocity in the case of the rough surfaces. The air is supposed to be made up of positive and negative elements called 'ions,' which are joined together by mutual attraction. It requires a considerable force to separate them and thus produce an electrical movement or current, but when this is once done and there is a rapid movement these ions are hurried past one another without being able to unite or become locked together again.

In order to make this arc the means of applying a force to a combination of a condenser and inductance, we have only to arrange, as in Fig. 146, to supply it with current from a steady source like a battery or D.C. generator, and to apply the P.D. at its terminals to a condenser and choking coil in series. This arrangement is the analogue of Fig. 147, in which the hairs of a violin bow have been arranged as a continuous cord round some pulleys to form a complete mechanical circuit. This cord is kept in continuous rotation by a motor like the continuous flow of current round the arc-lamp circuit, and a violin string is pressed against the cord. The result is to get a continuous note or vibration from the violin, and we ought to expect a continuous electrical oscillation in our condenser and inductance. The existence of these oscillations was first shown by Mr. Duddell in 1900, and they are most clearly manifested owing to the fact that the arc acts as a telephone, and the oscillations passing through it from the condenser and inductance produce sound waves of the same frequency. The arc therefore sings or whistles with a continuous note, which corresponds to the oscillation frequency of the combination of capacity and inductance.

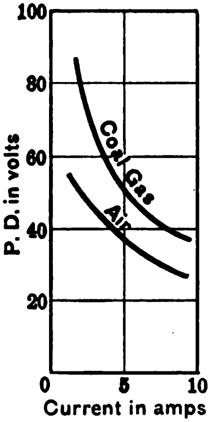

FIG. 145.—Poulsen arc characteristic. Length of arc, 2 mm.

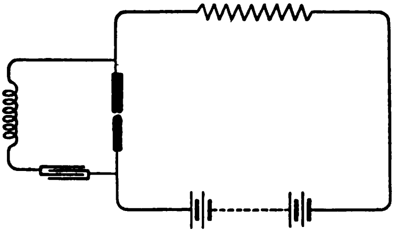

FIG. 146.—Electrical circuit for continuous oscillations.

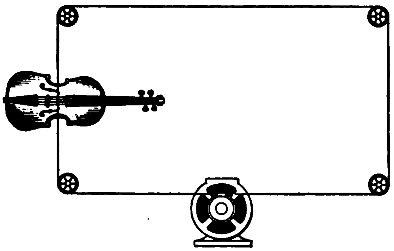

FIG. 147.—Mechanical circuit for continuous oscillations.

The actions which take place in this case are readily understood. Suppose that the arc is burning at first steadily without connection to the oscillating circuit, and that there is sufficient resistance or inductance in series with it to prevent the total current changing much if the resistance of the arc alters. Now suppose that the condenser and inductance are suddenly connected to its terminals. The P.D. at the terminals of the arc will cause a sudden rush of current to charge the condenser, and the inductance will cause this current to continue and overcharge the condenser. This will also be aided by the fact that the moment the condenser is connected, and current flows into it, less current must pass through the arc. But owing to the characteristic of the arc this diminution of current causes a rise of P.D across the arc which helps to charge the condenser. When the overcharged condenser begins to discharge back through the arc, the current is increased and the P.D. consequently drops, which helps the discharge. Hence the arc helps the charge and discharge of the condenser and increases the amplitude of the oscillations like the violin bow. It is a curious coincidence that the electric arc, which received its name from its bow-like shape, should also have the same characteristics and function as the bow of a violin.

We may therefore sum up the mode of stimulation of mechanical or musical and electrical vibrations as follows:

Nature of Stimulus.	Mechanical or Musical.	Electrical.
Impulsive	*Pizzicato*	Spark
Continuous	*Arco*	Arc

It is evident that the same comparison holds between the spark and arc methods of electrical stimulus as between the *pizzicato* and bow modes of vibrating the string, *i.e.* the former produces sets or trains of damped vibrations, and the latter continuous forced vibrations. As electrical vibrations are generally of the order of hundreds of thousands or millions per second, and it is not easy to produce sparks at more than a few hundreds a second, there are huge gaps between the successive sets of oscillations, and this has led to the introduction of arc methods. But the forced or non-harmonic character of these oscillations has made itself apparent, and has been a source of considerable trouble wherever perfect tuning or resonance is required, as in wireless telephony. For this reason several experimenters have abandoned oscillations altogether, and have built special alternators to give actual high-frequency alternating currents.

Properties of High-Frequency Currents.—We will now very briefly refer to the effects of high-frequency currents. The most remarkable of these are their behaviour as regards inductive circuits, their physiological effects, and the actions which take place in the surrounding space.

As regards inductive effects, it is evident that at high frequencies reactance becomes relatively very important. At a frequency of 1,000,000 periods a henry of inductance is equivalent to $2\pi nL = 6{\cdot}28$ megohms of resistance, so that a coil of a few millihenries has a reactance of tens of thousands of ohms. As a result we often find that the discharge will spark across an air-gap, when shunted by a loop of thick copper. Fig. 148 shows an experiment which illustrates this very effectively. Two ordinary Leyden jars are connected to an induction coil, provided with a discharger, and their outer coatings are connected by a loop of copper rod, say, $\frac{1}{4}''$ diameter and 6 or 8 feet long. Every time a spark passes between the discharger knobs, high-frequency oscillations are set up in the jars and loop. If, however, the two ends of the loop are brought close together as shown, the voltage between the ends is so high, although the resistance is only a few thousandths of an ohm, that a discharge will take place across the gap every time an exciting spark passes. A straight-filament glow-lamp connected across the loop will glow, but if an ordinary loop

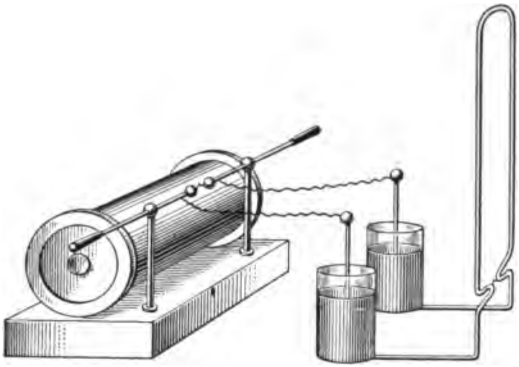

FIG. 148.—Alternative path experiment.

filament is used a discharge will pass between its terminals rather than round the curled filament.

Of the physiological effects it is unnecessary to speak here, except that it may be mentioned that much greater currents can be passed through the body without injury than with direct currents or alternate currents of low frequency.

Turning, however, to the effects external to the circuit, we enter upon a most important and fascinating field, concerning which we can only point out a few of the leading phenomena. It is evident that as these high-frequency currents are accompanied by a magnetic field surrounding the conductor, they will produce strong mutual induction effects in neighbouring circuits, especially when the other circuits have such capacity and inductance that the currents in them naturally tend to oscillate with the same frequency as that of the primary circuit or oscillator. In Fig. 149 we have an oscillating circuit consisting of a Leyden jar and a single loop of wire; and a short distance away a second similar jar and loop. Whenever a primary oscillation is set up, secondary oscillations are induced in the other circuit, and if the natural oscillation frequencies are equal, these oscillations will produce a high P.D. between the coatings of the jar. If a wire is brought over from the knob of the jar to nearly touch the outer coat, then each time that a spark passes in the primary circuit a flash takes place between the wire and the outer coating of the second jar. The exact resonance of the secondary circuit is secured by a slider or bridge-piece which enables the area of the second loop to be altered, and it will be found that a very small alteration of the position of this bridge-piece makes the secondary sparks appear or disappear. This experiment, which is due to Sir Oliver Lodge, was termed by him the syntonic jars experiment, the word 'syntony' implying the equal tone or equal oscillation frequency of the two circuits.

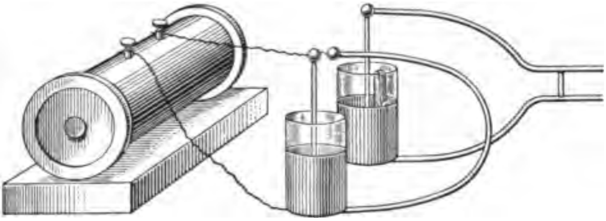

FIG. 149.—Syntonic jars.

If these circuits are made larger and are removed a considerable distance apart, and the secondary spark-gap is replaced by a sensitive detector or coherer, they correspond to a tuned wireless telegraphic transmitting and receiving station. We thus have an example of the transmission of electrical oscillations over a distance.

Now comes the interesting question as to the nature of the transmission of these oscillations. Let us take the case of an ordinary oscillator like the Hertz oscillator in Fig. 138. Suppose that one of these spheres is charged to a high potential. The result is that there are lines of electrostatic force radiating out from it in all directions. These lines of force, whatever their real nature, are known to behave like elastic threads, *i.e.* they try to shorten and get thicker, and thus produce an attraction between oppositely charged, and a repulsion between similarly charged bodies. Suppose that the P.D. is raised to such a point that a spark passes. The charge on the first sphere passes over to the second, carrying its lines of force with it, then rebounds and oscillates backwards and forwards. Now consider what happens when we have a long rope attached to a distant object, and we hold the end of it and oscillate our hand rapidly from side to side. The result is that waves or pulses are formed in the cord which travel away from the hand at a definite rate. The electrical oscillations are likewise accompanied by these transverse waves, which travel outwards at a definite rate, and which correspond to a certain extent to the sound waves emitted by a vibrating body. This was first predicted by Maxwell in 1876 from mathematical considerations, and he showed that the velocity of propagation should be the same as light, viz. 300 millions of metres per second in free space. This prediction of Maxwell's was completely verified by Hertz, and we now know not only that electromagnetic disturbances are propagated with the speed of light, but that radiant heat and light consist simply of electromagnetic waves of suitable frequency. If the oscillations are at the rate of 400 billions per second approximately they produce the sensation of red light, and if they are of about double this frequency or 800 billions per second they produce the sensation of violet light. These rates of oscillation correspond approximately to what should be obtained with circuits of the size of atoms, and the inference is that light is produced by electrical oscillations in the atoms of a luminous body and is transmitted as transverse ripples or waves in the space around. The waves employed in wireless telegraphy only differ from light in being of less frequency.

In dealing with electrical oscillations we have not taken into account the damping due to resistance, nor have we proved any of the statements we have just made as to the transmission of electromagnetic waves in space. These matters must be reserved for a later volume, but the slight sketch here given will help the reader to realise the importance of the subject.

CHAPTER XVI

SIMPLE HARMONIC MOTIONS IN DIFFERENT DIRECTIONS. POLYPHASE CURRENTS

We now come to a very important and interesting branch of the subject of alternate currents. In all our study of the composition of simple harmonic motions, and of alternate currents and potential differences, we have supposed that the motions took place in the same *direction* of space, that of the axis of the engine cylinder, or of the wire carrying the alternate currents. But if we take an ordinary simple pendulum composed of a string and a bob, we can give it a blow so as to set it swinging either north and south, east and west, or in any other direction we choose; we can also give it two impulses, one of which would set it swinging north and south and the other east and west. What will be the result?

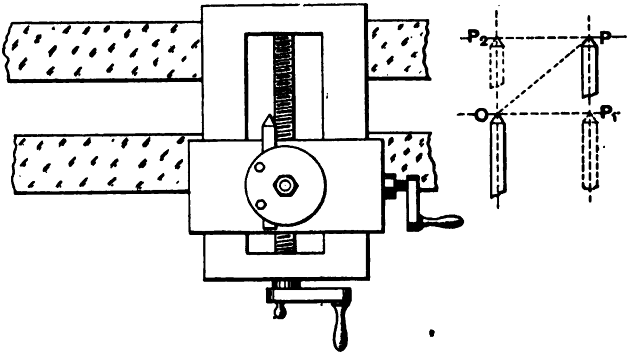

Fig. 150.—Motion of tool in slide rest.

Simple Harmonic Motions in Perpendicular Directions.—First suppose that we have two directions at right angles, like the two motions of a tool in the slide rest of a lathe (Fig. 150), along the bed and across it independently. We can move the screws in such a way that either movement separately would give the tool a simple harmonic motion.

Now if both these motions are carried on simultaneously it is clear that they are perfectly independent of one another, and that the tool will therefore have reached a position at each instant which is exactly as far in each of the two directions as if the motion in that direction had alone existed. That is to say, that if the tool was originally at the point O, and receives a motion along the bed of the lathe which would have carried it a distance x to the point P_1; and, simultaneously, another motion across the bed which would have moved it a distance y to the point P_2; it will really reach the point P, which is at the distance x laterally and y forward from O. In other words, it will have moved along the resultant of the two displacements OP_1 and OP_2.

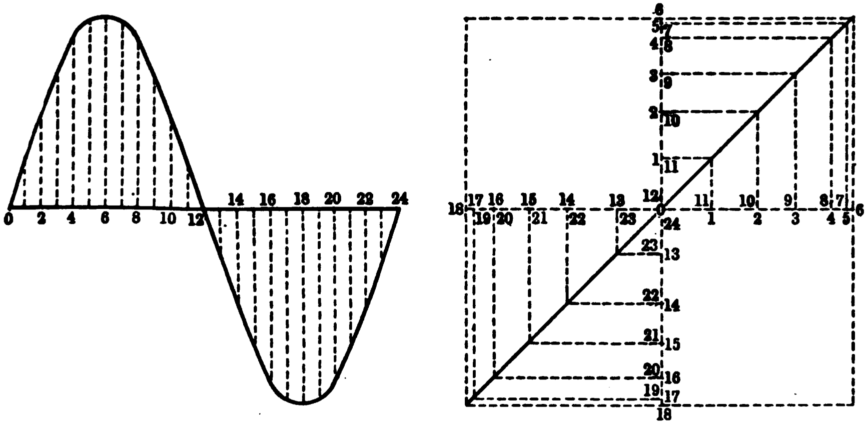

FIG. 151.—Two simple harmonic motions in perpendicular directions in phase.

We will now suppose that each of these motions varies in a simple harmonic manner, and first that they have each the same frequency, amplitude, and phase; in fact, that they are both represented by the same sine curve as in Fig. 151. If we divide the whole period up into 24 equal intervals of time, then the lateral and forward displacements at each instant are given by the height of the curve at that instant. Hence if we number the intervals from 1 to 24 we can mark off the distance $O1$, $O2$, $O3$, &c., along each of the two directions at right angles; and if we then complete the parallelogram at each time, we get a series of points, representing the position of the tool at each instant. But since the horizontal and vertical distances are both equal, the total displacement of the tool at each instant will be $\sqrt{2}$ times either of the component displacements and will be inclined at 45° to either of them. It is evident, therefore, that the motion of the tool is along a straight line at 45°, and is a simple harmonic motion of the same frequency and phase as the original motions, but of $\sqrt{2}$ times the amplitude.

Next suppose that the two motions have the same frequency and amplitude, but that the traversing or cross motion lags one interval behind the longitudinal motion, which corresponds to a lag of $\frac{360}{24}$ or 15° of phase angle. The result is evidently that when the longitudinal motion is given by $O1$ the transverse motion is zero, when the longitudinal has reached 2 the transverse has only got to 1, and so on. Completing the parallelograms as before and drawing a curve through the points so obtained, we find that an oval or elliptic curve is obtained which represents the motion of the tool (Fig. 152).

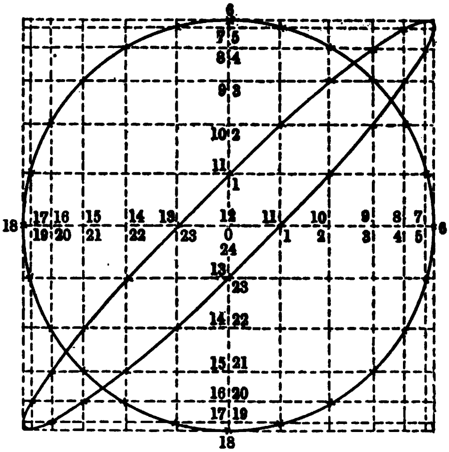

FIG. 152.—Two simple harmonic motions in perpendicular directions. Difference of phase 15° and 90°.

If we do the same thing with a lag of two intervals or 30°, we find that we get an oval curve as before, but broader and shorter, the greatest length being in one direction at 45°, and the least at 45° in the other direction; and this process goes on further and further until, when the difference of phase is 90°, the oval is equally long and broad and appears to be circular in form. If the lag still further increases, the length in the first direction continues to decrease and the breadth to increase, so that the breadth is greater than the length, or the oval has its longer axis in the opposite direction (Fig. 153). This process goes on with increasing lag until the oval gradually gets longer in the new direction, and narrower until it finally shuts up and becomes a straight line at 45° when the lag is 180°. This is obviously correct, as when the lag is 180° the conditions are exactly the same as when the two motions are in phase (Fig. 151), except that the transverse displacement at each instant is precisely opposite to its former amount.

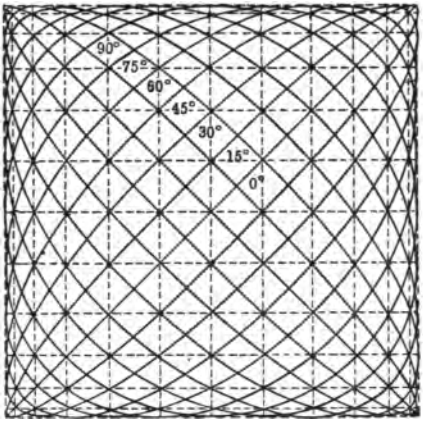

FIG. 153.—Two simple harmonic motions in perpendicular directions. Difference of phase 15°, 30°, &c.

One of the simplest ways of realising this is to take a plain ring of wire and hold it in front of one with the two hands at the opposite ends of a diameter inclined at 45° to the horizontal. If the hands are then approached nearer to one another the ring becomes an oval, and finally shuts up into a straight line at 45°. If, on the other hand, the hands are moved farther and farther apart, the circle turns to an oval with its long axis at 45° in the opposite direction, and finally shuts up into a straight line again. The length of this line is, however, $\frac{\pi}{2}=1{\cdot}57$ times the diameter instead of $\sqrt{2}$ or $1{\cdot}41$ times the diameter of the circle in the case of the simple harmonic motions.

We therefore see that the usual result of imparting two simple harmonic motions to a body of equal amplitude and frequency is to make it move in an oval curve, and that when the phases are coincident or 180° apart the oval becomes a straight line, while when they are 90° apart, it appears to be circular. We can verify this approximately in an instructive way by hanging up a weight by a piece of string to serve as a pendulum, and striking it with the hand or some form of bat. After a little practice one can learn to give it an impulse in any direction of practically always the same amount.

Now give the pendulum a blow that sets it swinging from side to side, and, when it is swinging, try to give it an exactly equal blow in a perpendicular direction, just as it passes the middle of its swing. The result will be that it starts off in both directions simultaneously, or as if it had two simple harmonic motions in perpendicular directions in phase with one another; and if the force of the second blow and the moment of striking have been just right, the pendulum will swing in a straight line at 45°, as in Fig. 154. On the other hand, if the pendulum has been set swinging from side to side as before, and if the second impulse in the other direction is given, not when it passes the middle of its swing, but a quarter of a period later, just when it reaches the end of its motion, it will be found to move in a circle, as in Fig. 155. In all other cases the pendulum will describe an oval path.

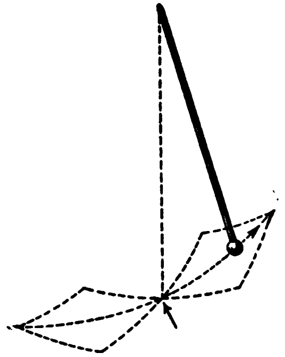

FIG. 154.—Motion of pendulum with two perpendicular impulses in phase.

The most important case for our present purpose is that when the two motions differ in phase by 90°. In Chapter VII, in which the mean and the effective values of simple harmonic quantities were examined, it has already (on p. 98) been noticed that when a crank rotated uniformly, and was attached to both a horizontal and a vertical piston, each of these pistons moved with a simple harmonic motion, but one of them lagged a quarter period or 90° behind the other; for, when the crank was horizontal, the vertical piston was at the centre, and the horizontal piston at the end, of its stroke. The amplitude and frequency of the two motions were evidently equal. Conversely, if we were to give each of the two pistons a simple harmonic motion of equal frequency and amplitude, but differing 90° in phase, the crank pin would rotate uniformly round a circle even if there were no crank shaft to constrain it.

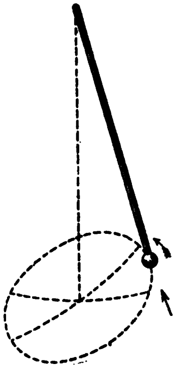

FIG. 155.—Motion of pendulum with two perpendicular impulses differing 90° in phase.

An effective way of showing the composition of these motions is by the simple apparatus illustrated in Fig. 156. It consists of a vertical board on which are fixed four pulleys. Two cords move round these pulleys, one of which moves horizontally along the board and the other vertically up and down it. The two cords are attached together at a

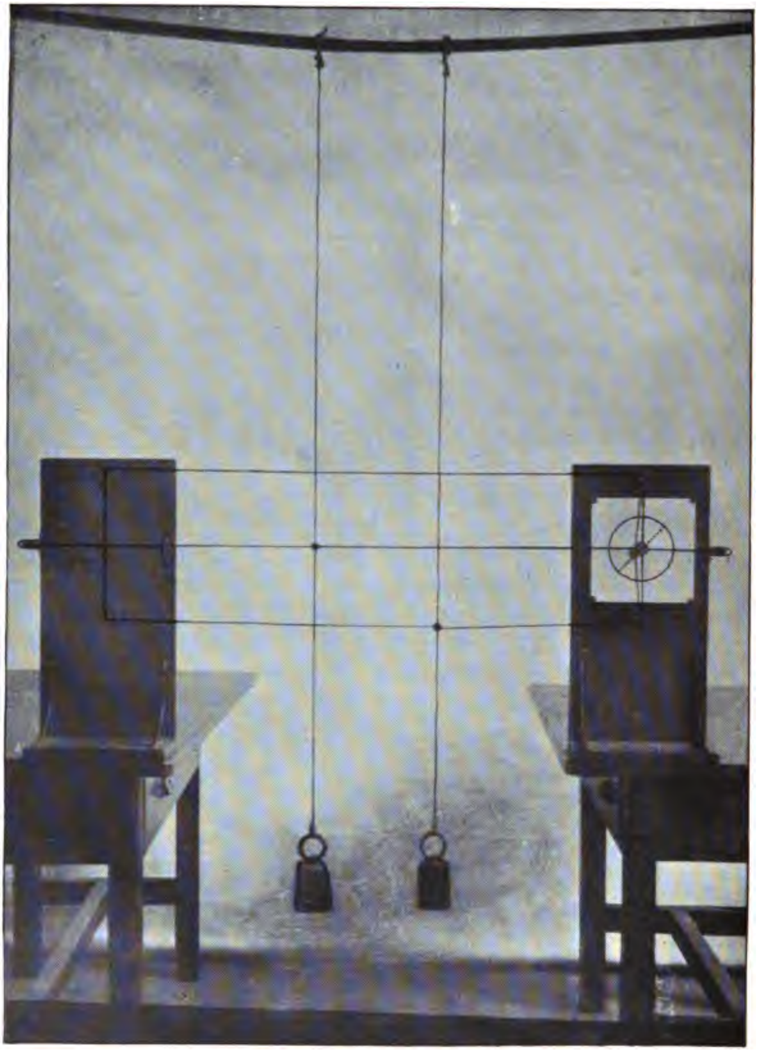

FIG. 156.—Apparatus for compounding two perpendicular harmonic motions.

point, and a pencil is fixed at their junction which moves over a sheet of paper attached to the board. Two long similar heavy pendulums are suspended side by side from the ceiling, and each is joined to one of the above-mentioned cords. If pendulum A is set in motion, it imparts a simple harmonic motion to the pencil along a horizontal line; while if pendulum B swings, the pencil moves with a simple harmonic motion of the same frequency but in a vertical direction. If now the two pendulums are set swinging together, with the same length of swing, the pencil marks out a straight line at 45°; and if one pendulum is at the end of its swing while the other is at the centre, the pencil traces out a circle as shown.

We can now collect the above facts into the following statement:

Two simple harmonic motions of equal frequency and amplitude in directions at right angles to one another combine into an oval or elliptic motion, which becomes a simple harmonic motion along a straight line inclined at 45° to the original directions, if the two motions are in phase or have 180° difference in phase; and becomes a uniform circular motion if they have 90° difference of phase.

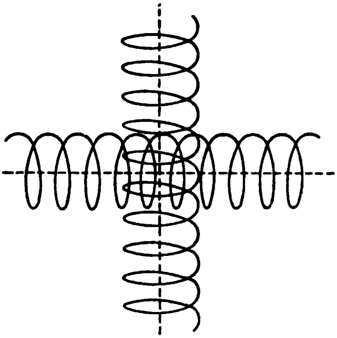

FIG. 157.—Two perpendicular coils for combining magnetic fields.

Rotating Magnetic Fields.—So far we have been dealing with ordinary mechanical motions. Let us now take the case of two coils of wire wound at right angles to one another, as in Fig. 157, having equal diameters and numbers of turns. If an alternating current is passed through one of them, there is produced an alternating magnetic field along the direction of the axis of the coil. If the same alternating current is passed through both, a resultant alternating field at 45° to the axes of the coils is produced. But if instead of this we pass two separate alternating currents through the two coils, of the same strength and frequency but differing 90° in phase, we obtain a magnetic field which has always the same strength but which rotates uniformly, the number of revolutions of the field per second being equal to the frequency.

A very convenient method of studying these phenomena is by an apparatus like that shown in Fig. 158.* It consists of a disc of wood or slate, having holes in it through which is wound bare resistance wire to form a re-entrant winding, precisely like a Gramme ring armature. Two diametral points on this winding are brought down to slip-rings on the shaft which carries the disc, and the brushes pressing on these slip-rings are connected to an accumulator battery. The result is that a current continually flows along the two halves of the winding and there is a steady fall of potential from the point of entry of the current to that of leaving. If two brushes B_1B_2 press on the wire at opposite ends of a diameter and the disc is rotated, it is evident that an alternating P.D. is obtained between these brushes of frequency equal to the rate of rotation of the disc. The variation will not be of simple harmonic form unless the winding is suitably graduated, but this does not much matter for our present purpose. Now if, in addition to our first pair of brushes, there is a second pair B_3B_4 touching along a diameter perpendicular to the first, it is obvious that this new pair will have a P.D. which has the same amplitude and frequency as that between the first pair, but which has a difference of phase of 90°, since the disc makes a quarter of a turn in moving any point of its winding from a brush of the first pair to one of the second.

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* An apparatus of this kind was constructed and shown by the author at a meeting of the students of the Institution of Electrical Engineers in 1894.

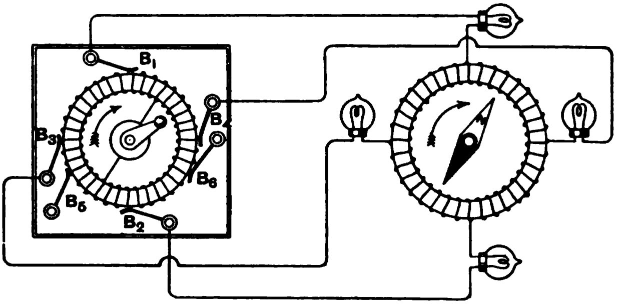

FIG. 158.—Hand commutator, and device to show rotating field.

Now if we take a ring of iron wire and wind it with insulated wire, also like a Gramme ring as shown on the right of Fig. 158, we know that a current passed between two ends of a diameter of the ring will divide into two parts along the two sides of the ring, and will magnetise the core with consequent poles just opposite the points of entry and exit of the current. A magnetic needle pivoted in the centre of the ring therefore points along the line joining these points of entry and exit. If instead of two tappings we have four, as in Fig. 158, and connect them to the four brushes of our commutator and turn the disc steadily, we have the two alternating currents differing 90° in phase passing through the windings in two directions at right angles; we thus get a uniformly rotating magnetic field, and the magnetic needle turns round at the same rate as the disc. If at the same time we put glow-lamps in series with the leads they glow and go out as the currents rise and fall, and one pair of lamps are at their full brightness when the other pair are extinguished, and *vice versa*.

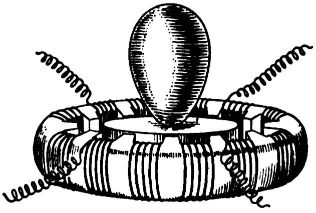

FIG. 159.—Prof. S. P. Thompson's experiment of rotating copper egg.

If the same thing is done with a larger ring and two-phase currents of considerable strength and ordinary frequency, we get very powerful rapidly rotating magnetic fields which show most interesting effects. Such an arrangement was first used by Prof. S. P. Thompson, the ring being provided with internal projecting poles, as in Fig. 159. With currents of the ordinary frequency of 50 ~ per sec., the field rotates 50 times per sec. or 3000 turns per minute, and this is too fast for a magnetic needle to follow unless it is previously spun up to speed mechanically. But if we put any mass of metal in the field, currents will be induced in it by the motion of the field, and this will tend to cause the mass to rotate with the field. This is easily seen, as if the metal is turning at the same speed as the field the lines of force do not cut it, and no current is induced. But if it moves more slowly than the field, the magnetic lines gain or move round the mass and induce currents in it which are directed as shown in Fig. 160. Whenever a conductor carrying a current is in a magnetic field, there is a magnetic drag which is perpendicular to the current and to the field, and thus produces a force in the direction of the rotation. The conducting mass, therefore, if perfectly free to rotate, tends to get up speed in the direction of

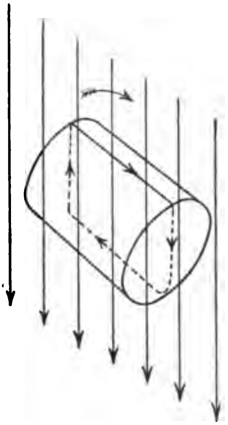

FIG. 160.—Generation of eddy currents in cylinder by rotating magnetic field.

rotation of the field, until it nearly reaches the speed of the field or what is called the synchronous speed; but it always moves somewhat slower than the field in order to have enough induced current to overcome the force of friction. If the retarding force is increased, the speed decreases somewhat until the increase of the induced currents balances the greater torque. The amount by which the speed of rotation is lower than that of the field or the synchronous speed is called the *slip*, and the slip therefore increases as the torque is increased.

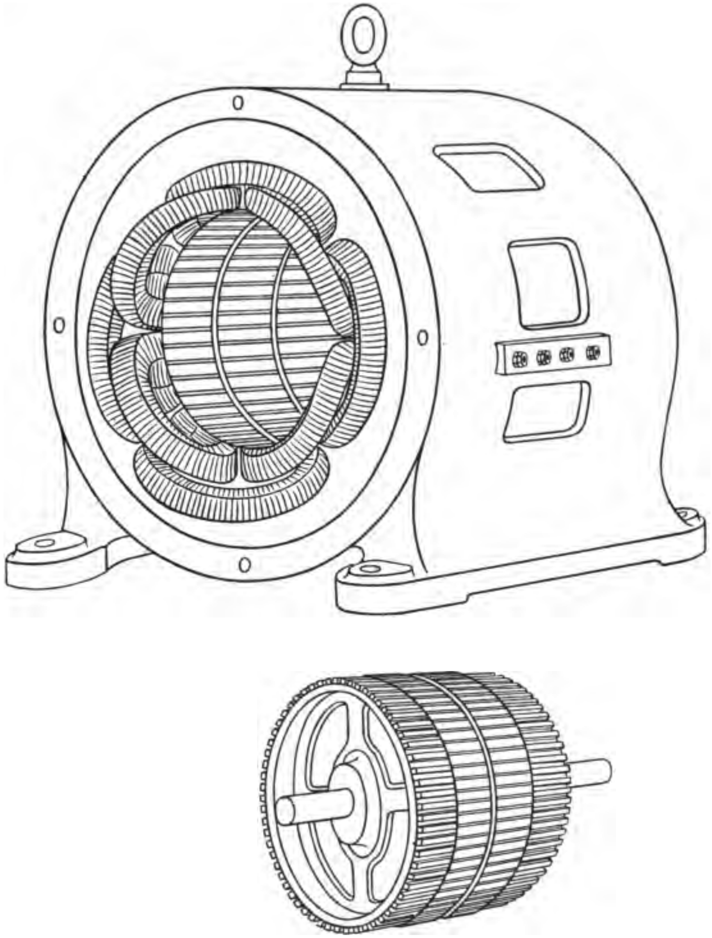

FIG. 161.—Two-phase induction motor.

Thus the result of placing such a metallic mass in the rotating field is that it begins to spin vigorously and runs rapidly up to nearly synchronous speed. A most effective way of showing this is by making use of a large hollow egg of sheet copper. If this is placed on a plate in the magnetic field it first begins to spin round on its side, but as the speed rises it commences to wobble, and finally stands right up and spins quietly on its small end.

In order to make a self-starting alternate-current motor, therefore, all that is necessary is to have an outer iron ring or *stator*, and an inner revolving portion or *rotor* (Fig. 161). The stator can be wound with an ordinary ring or drum winding or with two sets of coils, while the rotor is most suitably made with an iron core to concentrate the magnetic flux, and with copper bars in holes or slots at the periphery. These copper bars may all be short-circuited together at the ends, giving us a short-circuited or squirrel-cage rotor. The two-phase squirrel-cage alternate-current motor is therefore the simplest motor in existence, as there is absolutely no electric connection between the revolving and the stationary part, and it is very efficient and robust. For improving the starting and regulation the rotor is, however, often wound with an insulated winding, and connected to three slip-rings, but the reasons for this will not be gone into here.

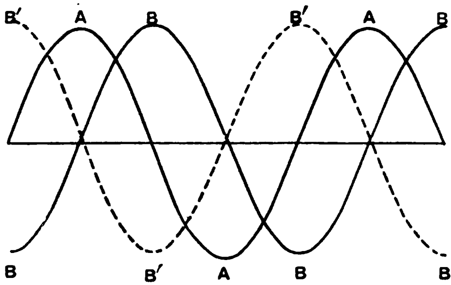

FIG. 162.—Effect of reversing one phase of a two-phase supply.

Reversal of the Direction of Rotation.—It will be readily seen, from the fundamental principles of combining simple harmonic motions in directions at right angles, that if one of these motions is reversed in direction or phase, the direction of rotation is also reversed. If, for example, in the case of our pendulum in Fig. 155 the second blow had been delivered in the opposite direction, it is evident that the pendulum would swing in a circle as before but in the opposite direction. Hence in a two-phase motor, if we reverse either pair of the supply leads the motor runs in the opposite direction. It seems curious to speak of the reversal of an alternating current which is always reversing, but as seen in Fig. 162, whenever we have two alternating currents, and we reverse one of them, it makes all the difference as to whether it is lagging or leading.

A beautiful method of showing the properties of two-phase currents and rotating fields is by means of the simple piece of apparatus shown in Fig. 163. It consists simply of two small horse-shoe magnets with iron cores about ¼″ diameter made of thin iron wire. Over these cores ordinary electric-bell bobbins are slipped and connected so as to magnetise the cores in the ordinary way. The two magnets are put together so that their poles are at the four corners of a square.

Between these poles is a single vertical piece of soft iron wire which is surrounded by another bobbin but has space to move about 2 mm. in any direction, and this iron wire bears a small mirror at the top. A small steady current is passed round the central bobbin, thus polarising the iron wire. If now an alternating current is passed round the coils of one of the magnets, the polarised iron wire is attracted rapidly backwards and forwards and a horizontal streak of light is thrown on the screen. On the other hand, if the alternating current passes through the other two coils, the iron wire moves in a direction at right angles, and a vertical streak of light is produced. If two alternate currents of the same frequency and magnitude are passed through these circuits and in phase with one another, a streak of light at 45° to the horizontal is obtained. On causing one of the currents to lag or lead by introducing inductance or capacity or otherwise, this streak opens out into an ellipse which becomes wider as the phase angle increases and attains the circular form when the currents are in quadrature. On still further increasing the phase difference an ellipse is formed

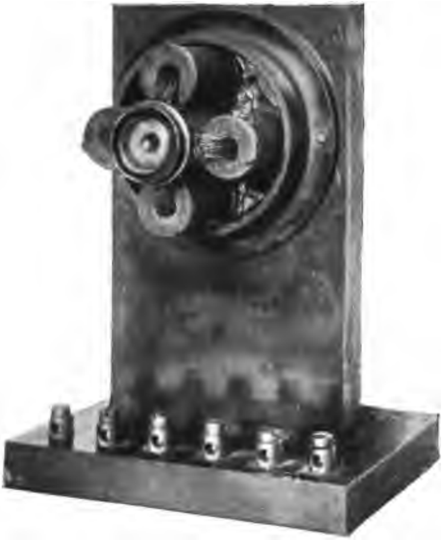

FIG. 163.—Rotating field oscillograph.

with its axes in the opposite direction, and finally a streak of light at 45° on the other side of the vertical. Fig. 164 shows photographs of curves taken in this way, which should be compared with Fig. 153.

Production of Two-Phase Currents.—There are several ways of producing two-phase currents, which are all easily understood. One would, of course, be to have two similar alternators with their shafts coupled together at such an angle that one was giving its maximum E.M.F. while the other was giving none. But if we have an ordinary single-phase alternator consisting of a field magnet revolving inside a stationary armature, wound as shown in Fig. 165, a second similar series of coils can be wound between the first series. Since a complete period corresponds to a movement of the field magnet from one north pole to another or over two poles, it will be seen that as the new winding is displaced by half a pole-pitch, its E.M.F. will be displaced by a quarter of a period or 90° in phase. The second winding does not involve any alteration of the first one, so that a single-phase alternator can be converted into a two-phase generator, giving approximately double the output without any modification beyond the provision of

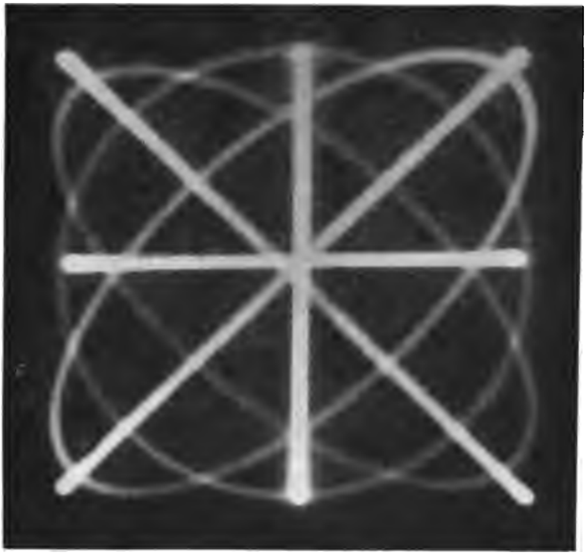

FIG. 164.—Curves taken with rotating field oscillograph.

additional slots in the iron core for the reception of the second winding.

A third method is also of considerable interest. If we take any two-pole direct-current dynamo, which is wound as a Gramme ring or the equivalent drum winding, and connect opposite points of this armature to two slip-rings, we evidently have an alternating E.M.F. between these rings, each period corresponding to a revolution of the armature. Now if we have two other tappings to two points on the armature midway between the first two, also connected to slip-rings, so that the four tappings divide the armature winding into four equal and symmetrical parts, we must have an equal alternating E.M.F. between these two rings, but lagging or leading by 90°. Such a machine, having an ordinary direct-current armature and commutator and four slip-rings, can be used in a variety of ways. It can be turned mechanically by some form of motor, and it then becomes a self-excited direct-current generator, and also a self-excited single-phase or two-phase alternator. It may be run as a motor from a direct-current supply, and it will then give single or two-phase alternate currents from its slip-rings. Lastly, it may be supplied with single-phase or two-phase alternating currents, and if then brought up to speed it will continue to run as a synchronous alternate-current motor, and at the same time will excite itself and give out direct-current power from the brushes on the commutator. In the latter case it is known as a rotary converter, and rotary converters are in considerable use for transforming two-phase or three-phase alternating currents to direct currents, especially for electric railway work.

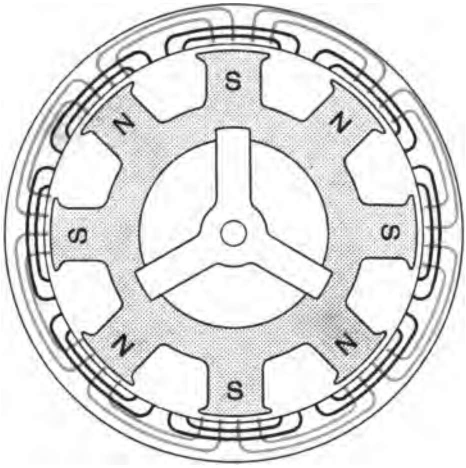

FIG. 165.—Diagrammatic representation of two-phase alternator.

Phase Splitting.—Another way of producing two-phase currents is obviously by making use of the differences of phase caused by inductances or condensers, and thus they can be obtained from a single-phase supply. If a resistance and inductance are connected in parallel, the current in the former is in phase with the P.D., while if the inductive coil has no appreciable resistance the current in it will lag 90° behind the P.D. Consequently, if the reactance of the coil is equal to the resistance of the other circuit, we shall have two currents of equal magnitude differing by 90° in the two circuits, or a two-phase supply derived from the single-phase mains. Again, if the inductance is replaced by a suitable condenser, a current leading instead of lagging 90° is obtained. It would appear, therefore, that we can always obtain a two-phase supply from single-phase mains, by employing two circuits in parallel, one of which is simply a resistance, and the other a choking coil or condenser. This is perfectly true so long as we do not make any use of this supply, but if we connect the two coils of a two-phase motor in the two circuits, it must be remembered that these coils have both resistance and inductance, and the accuracy of the phase displacement is thereby upset. However, a very fair approach to a two-phase supply can be obtained by this method of what is called 'splitting the phase,' *i.e.* dividing up the current taken from the mains into two parts differing in phase from one another, and the result is to obtain not a pure rotating field in the motor but an oval or elliptical field, like one of those in Figs. 153 or 164, which, however, has quite enough turning power to start the motor. This is the

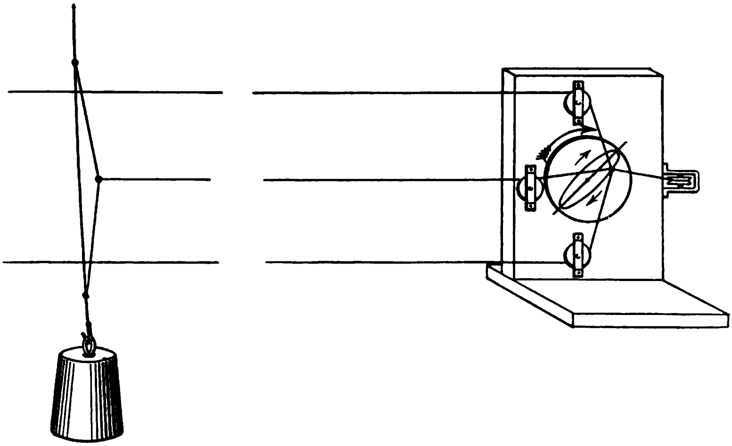

FIG. 166.—Phase splitting.

principle upon which the single-phase induction motor is based. These motors are nearly always wound as two-phase induction motors, and are started by splitting the phase in the manner above indicated. On reaching approximately full speed, however, one of the phases is usually disconnected and the motor continues to run with single-phase current in one of its circuits.

It is not proposed to fully explain the working of these or other motors here, but the following mechanical illustration will help to understand it. In Fig. 156 a board with two cords running over pulleys was shown, and these two cords could be moved by pendulums. Now, instead of the pencil fixed at the junction of the two cords writing simply upon a sheet of paper, suppose that there is a pivoted disc attached to the board (Fig. 166) which can turn freely and which represents the armature of the motor. If one of the cords is moved backwards and forwards the pencil will move in a straight line across a diameter of the disc, and however much it may rub against the disc it will not tend to turn it. If both cords are moved by the pendulums swinging together the pencil, as we saw before, will move across a diameter at 45° and will still not tend to turn the disc. But if there is the slightest difference of phase between the motions, the pencil will describe an oval curve and will evidently tend to turn the disc round. When the disc has got up a certain momentum, however, it will continue to rotate even if one of the cords is cut and the other one is only moved backwards and forwards. For the friction of the disc on the pencil will tend to carry it first to one side and then to the other of the axis, so that the forces in the cord help to keep up the motion. In other words, as soon as the disc or armature is in motion, it tends itself to produce an oval or rotating field as the momentum of a flywheel converts the reciprocating motion of a single-cylinder engine into a more or less uniform rotation.

The mechanical analogy to splitting the phase can be well shown by using the two cords and attaching them to one pendulum, which therefore gives a 'single-phase' motion. In this case the pencil simply moves over a diameter at 45° and produces no rotation. But now suppose that one of the cords remains fixed to the pendulum, while the other is attached to it through the medium of a piece of elastic or mechanical condenser, as shown in Fig. 166. The result will be a difference of phase in the motion of the two cords, and an oval motion of the pencil resulting in a starting of the disc.

Composition of Three Simple Harmonic Motions in Directions at 120°.—Supposing that we have a uniformly rotating crank as before, but that instead of connecting it to two pistons moving in cylinders at right angles to one another, we have three pistons in directions at 120° as in the Brotherhood three-cylinder engine (Fig. 167). As the crank rotates uniformly each of these pistons must move with a simple harmonic motion, and it is clear that each piston reaches the end of its stroke a third of a revolution later than the preceding one does. In other words, the motion of the pistons may be described as three simple harmonic motions of equal frequency and amplitude in directions at 120° and differing in phase by one-third of a period or 120°. Conversely, if the crank pin were disconnected from the crank so as to be free to move in any way, and the pistons were caused to move each with a simple harmonic motion of the same frequency and amplitude, but differing 120° in phase, the crank pin would move uniformly round a circle. Or we may say:

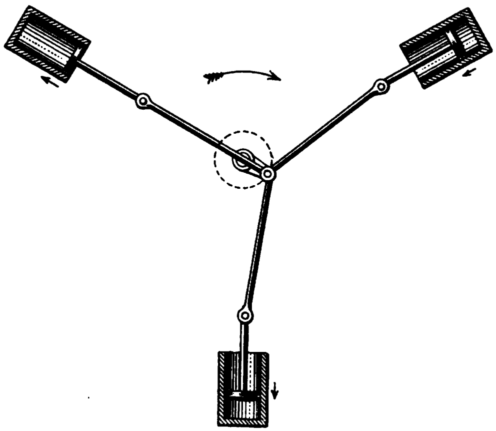

FIG. 167.—Brotherhood three-cylinder engine.

Three simple harmonic motions of equal frequency and amplitude in directions at 120°, *and differing* 120° *in phase, combine to produce a uniform circular motion.*

Three-Phase Alternate Currents.—It is easy to see from the above, and from what was said concerning the effects of two-phase currents, that if we have three solenoids or coils like those in Fig. 168, but intersecting one another at 120°, and supply them with three alternating currents of equal magnitude differing 120° in phase, the result will be to produce a uniformly rotating magnetic field, which will cause the rotation of a suitable rotor. But there is one important point which may be explained at once, before considering the utilisation of three-phase currents. In the case of two-phase supply there were two circuits involving the use of four leads or conductors. As a matter of fact, if these circuits are otherwise quite independent, two of the conductors can be combined into a common return with advantage, and we thus only require three wires for two-phase supply.* But when we come to three-phase currents, we have three circuits each with two terminals, and we therefore apparently require six leads. But if we examine Fig. 169, in which OC_1, OC_2, and OC_3 are the vectors of the three alternating currents, then the perpendiculars from the ends of these vectors upon the horizontal represent the magnitudes of the three currents in every position of the cranks. Now let us join three of the ends of the coils in Fig. 168

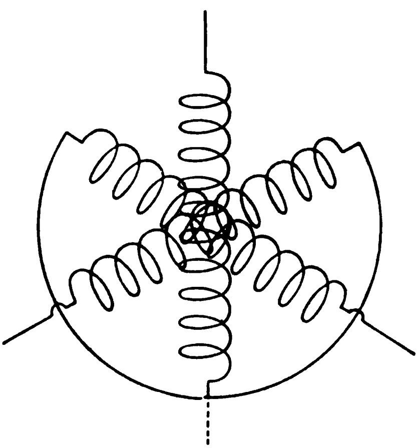

FIG. 168.—Combination of three circuits of three-phase system.

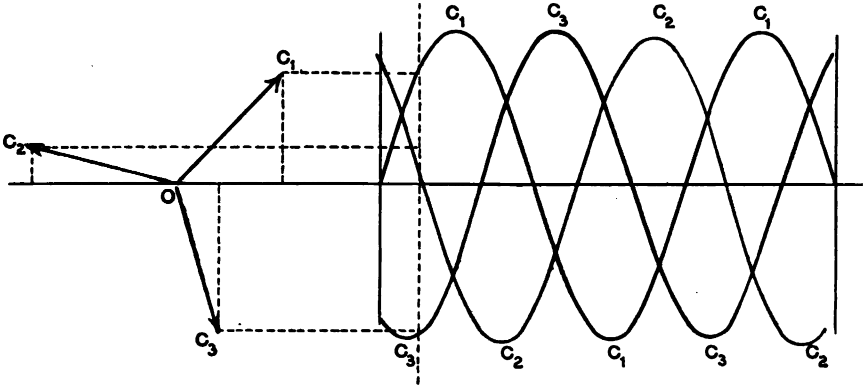

FIG. 169.—Vector representation of three-phase currents.

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* In the case of rotary converters this cannot be done, as the circuits are already connected through the armature coils, and a junction between any two leads would short-circuit a quarter of the armature winding.

together as shown. The current in a conductor from this junction must be the sum of the currents in the three other leads, or the sum of the three perpendiculars. But we saw in Chapter VIII that when adding simple harmonic currents together in the same conductor, we could add their vectors or cranks together to get the resultant vector or crank of the total current. If we do this in the present case, we first draw the vector OC_1, then add OC_2 to it, starting from the end of C_1, and finally add OC_3 by putting it on the end of C_2, and a vector from the beginning of the first to the end of the last vector should give us the vector of the resultant current. But the three vectors OC_1, OC_2, and OC_3 are equal, and they are equally inclined to one another, as they have equal phase differences. They therefore form a perfect equilateral triangle and the resultant is zero (Fig. 170). Hence if there is a conductor (shown dotted) from the junction point of Fig. 168, it carries no current, and it may therefore be dispensed with entirely and the coils supplied solely through the three leads. This result seems strange at first, but it simply means that the variation of the currents is such that, if two of them are positive or towards the junction, the third is negative or away from it and of amount equal to the sum of the other two; it therefore acts as a return to them without the necessity for an independent return lead.

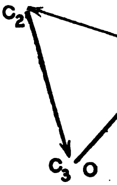


FIG. 170.—Vector sum of three-phase currents.

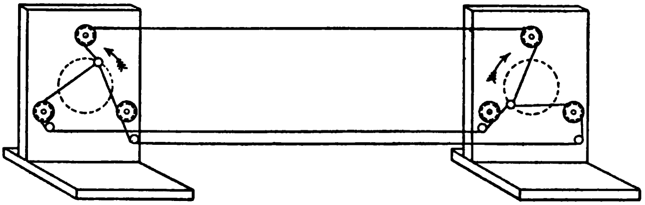


FIG. 171.—Mechanical model of three-phase working.

Either two-phase or three-phase supply can therefore be arranged with three leads, but in the latter case all three are equal. The turning moment in actual motors is somewhat more regular with two-phase than with three-phase supply, as is the case with three-crank in comparison with two-crank engines; and the economy in transmission can be made higher under suitable conditions. For these reasons three-phase supply is very commonly employed.

A fairly good mechanical analogy to a three-phase transmission by three wires is shown in Fig. 171, in which there are two upright boards each with three pulleys at the corners of an equilateral triangle, and three cords passing over the pulleys and attached together at each end. If the junction of the cords on the left is moved uniformly in a circle, it is evident that each of the cords will have a simple harmonic motion, and that the phase difference between these motions will be 120°. The combination of these motions communicates a circular motion to the junction of the cords on the right. This analogy, like that of the three-cylinder engine, is due to Prof. S. P. Thompson.

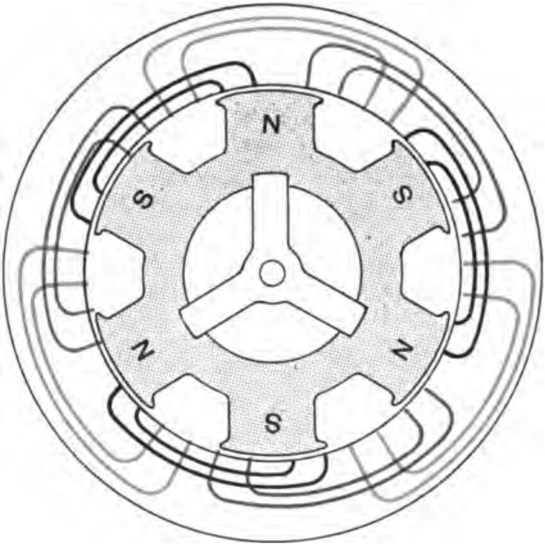


FIG. 172.—Three-phase winding for stator.

The three-phase motor is constructed exactly like the two-phase motor, except that the stator has three sets of coils displaced by one-third of a complete period (two poles), instead of two as in the two-phase motor. Fig. 172 shows a diagram of the connections of one form of three-phase motor.

Three-Phase Generation.—It is fairly evident that if we have an alternator with an ordinary revolving field and an armature winding corresponding exactly to the stator winding just given, we shall have three E.M.F.'s of equal magnitude displaced by

120°. If three ends of these circuits are connected together and the other three brought out to terminals, we have a star-connected three-phase generator, which may diagrammatically be represented as in Fig. 173. On the other hand, just as the algebraic sum of the currents in the three conductors of a three-phase supply was proved to be zero at every instant, so it can be shown that the algebraic sum of the voltages in the three circuits is zero at each instant. If, then, the three coils are connected together into a closed circuit or triangle, as in Fig. 174, no current will flow right round the circuit, but leads can be taken from the three corners of the triangle. This is known as triangle, or delta (Δ), or mesh connection, in distinction to the star or Y connection previously described.

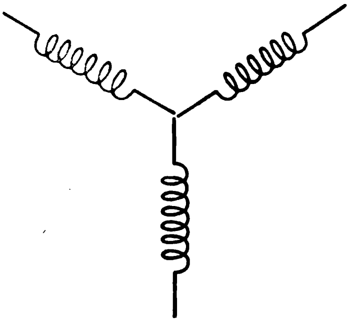


FIG. 173.—Star or Y connection.

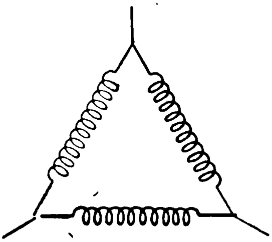


FIG. 174.—Delta (Δ) or mesh connection.

Voltages and Currents in Star or Mesh Connection.—Some little care has to be exercised in speaking of the P.D.'s or currents in a three-phase supply. If a generator has its three circuits connected in star to three leads, it is evident that the current in each lead is the same as the current in the coil to which it is attached. But when we come to the voltage, what are we to measure? The voltage of each coil is the P.D. between its outer end connected to the lead, and the middle or neutral point to which no lead is connected. The only voltages, therefore, which can be measured, away from the generator, are those between any two of the lines, which are evidently in each case the sum of the voltages of two of the coils together. These voltages are, however, not in phase, so they must be added as vectors. This means that if the effective voltages of the three circuits of the machines are each of amount v in Fig. 175, the voltages between lines $\bar{V}$ will be the sides of the triangle formed by joining the voltage vectors together. The sides of a right-angled triangle having an angle of 30° are in the ratio of 1, 2, and $\sqrt{3}$ (as seen from Fig. 176), hence half the voltage $\bar{V}$ or $\frac{\bar{V}}{2} = \frac{\sqrt{3}}{2}\bar{v}$, from which $\bar{V} = \sqrt{3}\,\bar{v}$. The voltages between lines are therefore $\sqrt{3}$ (or 1·732) times the voltage in the coils, and differ in phase from them by 30°.

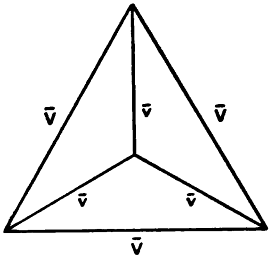


FIG. 175.—Voltages in phase and between lines.

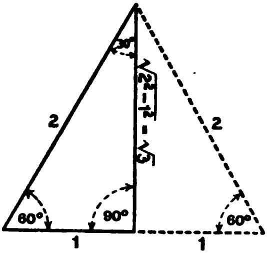


FIG. 176.—Ratio of sides in triangle with angles 30°, 60°, 90°.

On the other hand, if the coils are mesh- instead of star-connected, the conductors are joined to the ends of the coils, and hence the voltage between the lines equals the voltage of the coil, but the current in each lead is derived from two coils together. It is therefore the vector sum of the two currents, or $\bar{C}$, the line current, is $\sqrt{3}$ or 1·732 times the current in the coils.

When the P.D. and current of a three-phase supply are spoken of, it is always the P.D. between lines $\bar{V}$, and the current per line $\bar{C}$, which are implied unless otherwise specified.

Balanced and Unbalanced Loads.—A three-phase system should normally have not only the three voltages between lines equal, and differing 120° in phase, but also the three currents in the three conductors equal and differing in phase by the same amount. Where a three-phase generator supplies motors, this is practically always the case, as the three circuits of each motor should be almost exactly similar, and the load is spoken of as *balanced*. But if lamps are connected between the wires, as is sometimes done on the continent, there may be very different currents in the three conductors, and the system is then said to be *unbalanced*.

When the load is connected in mesh form, it is quite easy to calculate the currents in the lines. Suppose, for instance, as in Fig. 177, that we have a three-phase system at 220 volts (between lines) and 50 ∼, and that between lines 1 and 2 we have a resistance of 50 ohms, between lines 2 and 3 a choking coil of 20 ohms resistance and ·2 henry inductance, and between lines 3 and 1 a condenser of 30 microfarads capacity. Then we must have the following results:

Between lines 1 and 2, $\bar{c}_3 = \frac{V}{r} = \frac{220}{50} = 4{\cdot}4$ amps. in phase with P.D.

Between lines 2 and 3, $x = Lp = 314 \times {\cdot}2 = 62{\cdot}8$ ohms.

Hence $I = \sqrt{20^2 + 628^2} = 66$ ohms and $\tan \phi = \frac{62{\cdot}8}{20} = 3{\cdot}14$,

from which $\bar{c}_2 = \frac{220}{66} = 3{\cdot}34$ amps. $\overline{\diagdown}\, 72^\circ\ 21'$.

Between lines 3 and 1, $x = -\frac{1}{Kp} = -\frac{10^6}{30 \times 314} = -106$ ohms.

Hence $\bar{c}_1 = \frac{220}{106} = 2{\cdot}08$ amps. $\angle 90^\circ$.

The three current vectors may therefore be represented as in Fig. 178. But the conductor 1 carries the two currents $\bar{c}_2$ and

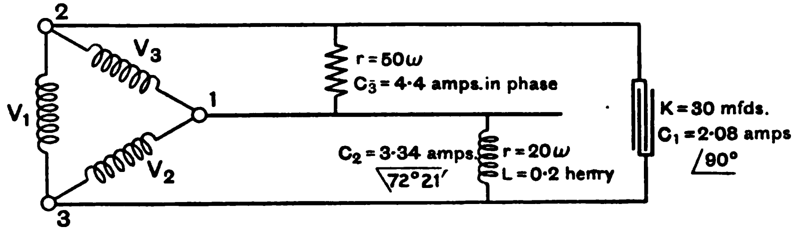


FIG. 177.—Unbalanced three-phase load.

$\bar{c}_3$. If these two currents were in phase with the voltages V_2 and V_3 respectively, the current $\bar{c}_2$ would be flowing *towards* the point 1, while the current $\bar{c}_3$ would be *away* from it. Hence the net current $\bar{c}_1$ supplied outwards from the machine at terminal 1 is the difference $\bar{c}_3 - \bar{c}_2$ and the vector $\hat{C}_1$ will be the difference between the vectors $\hat{c}_3$ and $\hat{c}_2$. Hence at the point 1 in Fig. 178(*b*) we draw the vector $\hat{c}_3$ and the vector $-\hat{c}_2$, equal and opposite to $\hat{c}_2$, and the diagonal of the parallelogram formed by $\hat{c}_3$ and $-\hat{c}_2$ is the current $\hat{C}_1$, which scales off as 7·65 amperes lagging 37° 40′ behind the phase voltage $\hat{v}_1$, which is radially outwards from the point 1. Similarly, at the terminal 2, the current $C_2 = c_1 - c_3$ at each instant, and $\hat{C}_2$ is the diagonal of the parallelogram formed by the vectors $\hat{c}_1$ and $-\hat{c}_3$. This gives us $\hat{C}_2$ as 6·27 amps. ∠ 39° 15′ in front of $\bar{v}_2$, and in like manner $\hat{C}_3$, which is the vector difference of $\hat{c}_2$ and $\hat{c}_1$, is 2·35 amps. lagging 142° 20′ behind $\hat{c}_3$. These three main currents $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$ are seen in Fig. 178 (*c*) to form a closed triangle, so that their sum is zero at every instant, as required by Kirchhoff's law.

When the load is connected in star instead of in mesh, it is somewhat more difficult to calculate the currents unless the load is balanced, and the discussion of this will be left for a later volume. With balanced loads, however, there is no difficulty, as in this case the potential of the junction of the three circuits is evidently the same as that of the neutral point of the star. The current in each branch will therefore be simply obtained by dividing the voltage between the line and neutral $\bar{v} = \dfrac{\bar{V}}{\sqrt{2}}$ by the impedance between the line and neutral point.

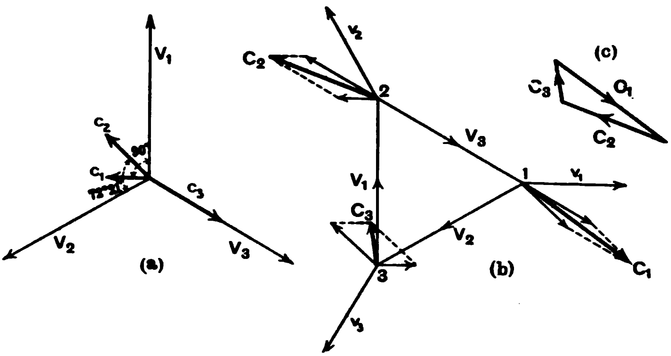


FIG. 178.—Unbalanced three-phase load.

Power in Polyphase Circuits.—There is little to be said about the measurement of power in two-phase circuits, as a two-phase system is simply equivalent to two single-phase circuits. It is therefore only necessary to use two wattmeters in these two circuits and to add their readings algebraically; while if it is known that the supply is balanced, *i.e.* that the P.D. and currents in both phases are exactly similar, a single wattmeter can be used in one circuit, and the reading doubled. It is rarely, however, that it is justifiable to do this when accuracy is required.

In the case of three-phase circuits the matter is somewhat more difficult at first sight, owing to the greater number of circuits and the inter-connection of the three phases. If the load is known to be balanced, the power in each phase is the same, and is evidently got by connecting a wattmeter with its main coil *M* in one lead and its shunt circuit *S* across one arm of the star, or between the lead and the neutral point. When at a distance from the generator, and when the neutral point is therefore not available, we can connect the shunt coil, as in Fig. 179, to the junction of two other coils R_1 and R_2 having the same resistance as the shunt circuit, and thus forming a star with it. In this case the wattmeter reading multiplied by 3 should give the total true power.

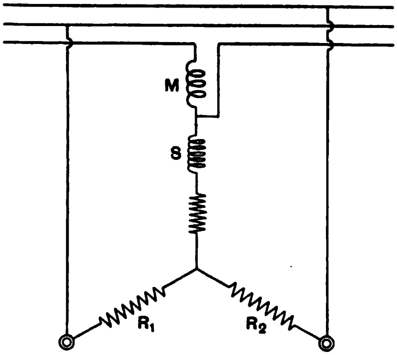


FIG. 179.—Measurement of power in balanced three-phase supply.

If the load is unbalanced, however, the above method is insufficient. It is evident that the total power would always be given by having three wattmeters, with their main coils in the three leads respectively, and with their shunts connected between these leads and the real or artificial neutral point, as in Fig. 180. In this case each wattmeter measures the power given to one arm of the star, and the sum of their readings is the total power. But this procedure is complex and costly, and it is consequently abandoned in favour of a method in which two wattmeters only are used, as follows:

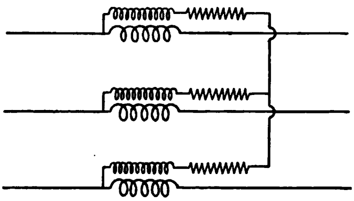


FIG. 180.—Measurement of three-phase power by three wattmeters.

In Fig. 181 let 1, 2, and 3 be three leads which may be carrying direct currents, or any alternating or unsteady currents whatsoever. If there are no other leads available we know that the algebraic sum of the currents $C_1 + C_2 + C_3$ must always be zero. If v_1, v_2, and v_3 are the voltages between the leads and the neutral point, then the total power at any instant $w = vC_1 + v_2C_2 + v_3C_3$. But since $C_1 + C_2 + C_3 = 0$, $C_3 = -(C_1 + C_2)$, and putting this in the expression for w we have—

$$w = v_1C_1 + v_2C_2 - v_3(C_1 + C_2) = (v_1 - v_3)C_1 + (v_2 - v_3)C_2$$

But v_1 is the amount by which the potential of line 1 is above that of the neutral point, while v_3 is the amount by which the potential of 3 is above that of the same point. Hence $v_1 - v_3 = V_2$, the amount by which the potential of lead 1 is above that of lead 3, and similarly $v_2 - v_3 = V_1$, the amount by which the potential of lead 2 is above that of lead 3. Hence we have—

$$w = V_2C_1 + V_1C_2$$

This is the power in watts at any instant if V_1, V_2, C_1 and C_2 are the values of the voltages and currents at that instant. Hence, if we have alternating currents the mean power in watts $\bar{w} =$ mean $V_2C_1 +$ mean V_1C_2. But if we connect a wattmeter with its main coil in lead 1, and its shunt connected between leads 1 and 3, its reading in watts $\bar{w}_1$ is evidently equal to mean V_2C_1, while if another wattmeter is connected with its main coil in lead 2 and shunt between 2 and 3, its reading $\bar{w}_2$ is similarly equal to mean V_1C_2. Hence we have $\bar{w} = \bar{w}_1 + \bar{w}_2$, or the true power in watts is the sum of the readings of the two wattmeters connected in this manner. This is true whatever the nature of the load, balanced or not.

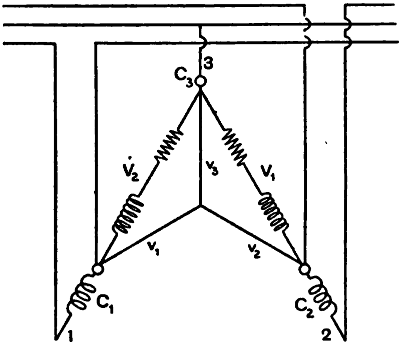
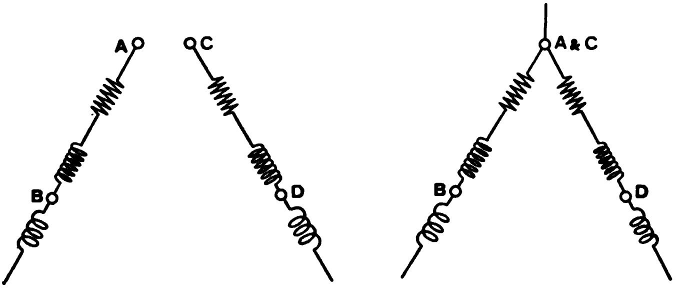


FIG. 181. — Two-wattmeter method of measuring power on three-phase unbalanced load.

As this matter is somewhat important, and is not always perfectly understood, the following explanation of it may be given: Suppose we have two pairs of conductors, AB and CD in Fig. 182, quite independent of one another, and each pair carrying current to a load. Now if a wattmeter is connected with its main coil in B and its shunt coil between A and B, it will measure the mean power in watts in the AB circuit. Similarly, a wattmeter with its main coil in D, and its shunt coil between C and D, would indicate the mean power in the

CD circuit, and the sum of the readings would be the total mean power in the two circuits, just as in the case of a two-phase supply. But if now leads *A* and *C* are combined together into a common conductor (as in Fig. 183), no change in the voltages, currents, or wattmeter readings will take place, and we then have three conductors with two wattmeters exactly as in Fig. 181. It is true that this does not correspond with a three-phase system, as we have loads only between the lead *AC* and *B*, and between *AC* and *D*, and none between *B* and *D*. But a little consideration will show that by suitably arranging the magnitudes and phases of the voltages and currents, we can get exactly the same conditions in the three leads as with an ordinary three-phase system; and the sum of the two wattmeter readings must therefore give the true mean power in every case, even if, for example, one circuit is fed with direct current and another with alternating current.



FIGS. 182 AND 183.—Explanation of two-wattmeter method.

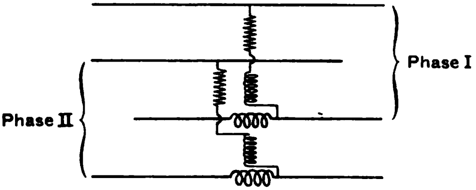


FIG. 184.—Two-wattmeter method for two-phase supply.

It is quite easy to make a wattmeter which will directly read the power on either single-phase, two-phase, or three-phase circuits, by having two fixed main coils and two shunt coils, the latter being mounted on the same shaft and swinging together. Each shunt coil must only be affected by the field due to its own main coil, and for that reason it is well to fix the two main coils with their axes at right angles, and similarly as regards the shunt coils. As the two shunt coils are on the same shaft, the torques due to them both are added together, and if the two portions are in every way equal, the total torque will be the sum of those which would be given by the two wattmeters separately. Such an instrument, if connected as in Fig. 184 for two-phase or Fig. 181 for three-phase circuits, will give the true power directly. A standard wattmeter for this purpose is shown in Fig. 185, and shows the two systems of main and shunt coils M_1, M_2 and S_1, S_2 respectively at right angles.

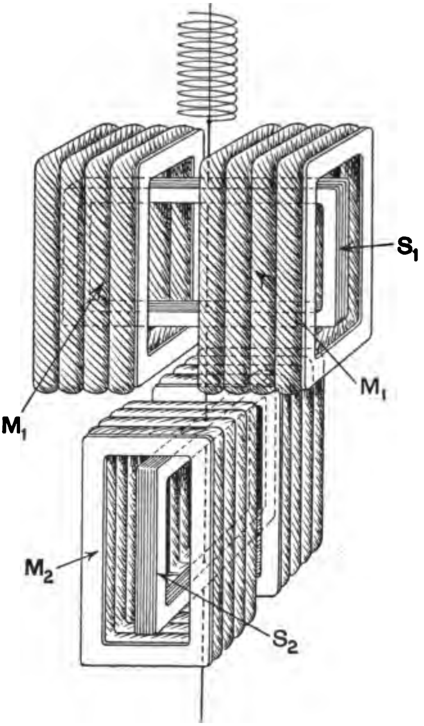


FIG. 185.—Standard double wattmeter.

Apparent Power and Power Factor on Three-phase Balanced Circuits.—If a three-phase system is balanced it is evident that the apparent power $\bar{W}$ is $3v\bar{C}$, as if we have a star-connected load the apparent power in each arm of the star is $v\bar{C}$, where v is the effective voltage on one arm of the star, or between each lead and the neutral point, and $\bar{C}$ is the effective current in each conductor. But we have seen that the voltage between lines

$$\bar{V} = \sqrt{3} \,.\, v, \text{ hence } v = \frac{\bar{V}}{\sqrt{3}} \text{ and } \bar{W} = 3\frac{\bar{V}}{\sqrt{3}}\bar{C} = \sqrt{3} \,.\, \bar{V}\bar{C}.$$

We thus have the following rule: *In a three-phase balanced system the apparent power in watts is got by taking* 1·732 *times the product of the effective voltage between lines and the effective current in amperes per line.*

To get the power factor we know that the true power in each arm of the star is $\bar{v}\bar{C} \cos \phi$ if the P.D. and current waves are sinusoidal or the total true power $\bar{w} = 3\bar{v}\bar{C} \cos \phi$. Since the total apparent power $\bar{W} = 3\bar{v}\bar{C}$, we have $\cos\phi = \frac{\bar{w}}{W}$ or true power divided by apparent power as with single-phase supply. But $\bar{w}$ is given by two ordinary wattmeters or a double wattmeter as above, and $W = \sqrt{3}\bar{V}\bar{C}$; hence $\cos\phi = \frac{\bar{w}}{\sqrt{3}\bar{V}\bar{C}}$, from which ϕ can be found. If the wave form is not sinusoidal, instead of $\cos\phi$ we speak of the power factor, as on p. 177, and then we have—

$$\text{Power factor } k = \frac{\bar{w}}{W} = \frac{\bar{w}}{\sqrt{3}\bar{V}\bar{C}} \text{ as above.}$$

If the load is out of balance, but only slightly so, we usually take $\bar{V}$ and $\bar{C}$ as the mean respectively of the three voltages between lines, and the three currents.

CHAPTER XVII

RECAPITULATION AND CONCLUSION

In the foregoing chapters we have dealt with the most essential portions of the elements of alternate-current theory, and we shall leave the more intricate questions of complex wave forms, the effect of iron, and of the propagation of impulses with and without conductors, for another volume. But, before closing, it will be well to survey the ground we have already travelled over, and to pick out the most important features of the reasoning, as this book will have failed in its aim if it does not enable any attentive reader not only to understand the properties of alternate currents, but to be so permeated with the scientific principles involved that he has a mental picture of what is taking place in any alternate-current problem, and is able to reproduce the reasoning and formulæ for himself at any time.

Glancing therefore over the previous chapters, we find that the work naturally divides up into four parts, as indicated in the text.

In Part I we began by comparing the circulation of a current in a circuit with that of water in pipes, or with the motion of a car, and we saw that the current resembled material motion in having a definite direction, in requiring force to maintain it, in experiencing opposition or friction to its motion, and in developing heat when moving against this friction. We therefore adapted mechanical terms to the electrical circuit, quantity of electricity being termed total electrical displacement in analogy to mechanical displacement;* current being spoken of as electrical velocity corresponding to mechanical velocity; and the rate of change of current as electrical acceleration corresponding to mechanical acceleration.

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* The term 'electric displacement' has unfortunately been employed in electrostatics, not for the *total* quantity of electricity displaced, but for the quantity *per unit area.* It would be preferable to use the term *displacement-density* for the latter in harmony with current-density.

Next dealing with the laws of the electrical circuit, we found that the motion of a body under the action of a force was affected by three quantities: the friction with which its motion is opposed, and also its mass or inertia which opposes its acceleration, and the elastic force of any spring control depending upon the displacement. We found that each of these quantities had its counterpart in the electrical circuit; that electrical resistance was the exact analogue of a mechanical friction in which the force was proportional to the speed of motion; that a coil of wire had a property known as self-induction or inductance which exactly corresponded with mechanical inertia in opposing the increase or decrease of a current with an electrical force proportional to the acceleration; and that a condenser behaves as an electrical spring in opposing an electrical displacement or quantity with a force proportional to that quantity. By allotting suitable units to resistance, inductance, and capacity, corresponding to the mechanical units of friction, mass, and the yielding of a spring, we obtained three electrical laws precisely corresponding to the three mechanical laws, as under:

Electrical.		Mechanical.	
Condenser	$V_1 = \frac{1}{K}Q$	Spring	$F_1 = \frac{1}{Y}s$
Resistance	$V_2 = rC$	Friction	$F_2 = fv$
Inductance	$V_3 = LA$	Inertia	$F_3 = ma$

The correspondence of these three laws led to our recognising that the variation of the current in an electrical circuit having any combination of resistance, inductance, and capacity under the action of any electrical force or voltage, could be exactly ascertained by finding how the velocity of a body varied when subjected to a mechanical force corresponding to the electrical voltage, and controlled by mechanical friction, inertia, and elasticity, exactly corresponding in amount to the resistance, inductance, and capacity of the circuit. In other words, the behaviour of the currents in any circuits can be exactly known from the velocity of a combination of a friction, a weight, and a spring, of suitable sizes arranged so as to correspond with the electrical circuits. This principle is the essential feature of our work, and its chief importance is that it enables us to visualise the phenomena of the electric circuit by comparing it with objects which are perfectly familiar and whose properties are readily realised. In Chapter IV other consequences of these analogies were deduced, notably as regards the waste or storage of energy, and certain well-known electrical formulæ were obtained with ease in this manner.

Our next step in Chapter V was to show how these simple principles allow us to find out what occurs when mechanical forces are applied to combinations of two or more of these mechanical quantities, friction, inertia, and elasticity; and thus, by analogy, what occurs when electrical force or voltage is applied to combinations of two or more of the electrical properties, resistance, inductance, and capacity. The correspondence of these results with actual experiment further justified the use of the mechanical analogies.

In Part I we therefore arrived at the stage of being able to find the force required to impart any motion to a combination of friction, inertia, and elasticity if the displacement, velocity, and acceleration of this motion were known; and similarly the voltage required to produce any current in a combination of resistance, inductance, and capacity if the electrical displacement, current, and acceleration were given. Our next business was to investigate the properties of the kind of motion corresponding to an alternating current, which we decided to be what is known as a simple harmonic motion, and to find its displacement, velocity, and acceleration, &c. In the first chapter of Part II, therefore, we explained the meaning of this simple harmonic motion and its widespread occurrence, and showed that it could most easily be pictured as the motion of the piston of an engine, attached by a long connecting rod to a uniformly turning crank.

Our next problem was to find the displacement, velocity, and acceleration of a body moving with a simple harmonic motion. We saw that the displacement was always given by the vertical component of the crank itself, while the velocity of the piston was the vertical component of the crank-pin velocity. This being known from the circumference of the crank circle and the rate of revolution, the results were arrived at that the velocity also varied in a simple harmonic manner, and that the maximum velocity was p times the maximum displacement and occurred a quarter of a period earlier, or $\hat{v} = p\hat{s} \angle 90^\circ$, where p was the angular velocity of the crank, or 2π times its frequency of rotation. From the fact that the velocity was the rate of change of the displacement, we decided that the rate of change of any simple harmonic quantity was simple harmonic, having an amplitude p times that of the original quantity and leading by 90°; and since the acceleration of the piston was the rate of change of its velocity, we immediately arrived at the relation $\hat{a} = p\hat{v} \angle 90°$, giving us the acceleration as varying in a simple harmonic manner and leading 90° in front of the velocity. Finally, as in alternate currents everything is expressed in terms of the current which corresponds to the velocity, we wrote the displacement and acceleration in terms of the velocity, and we then had $\hat{s} = \frac{\hat{v}}{p} \overline{\diagdown} 90°$, and $\hat{a} = p\hat{v} \angle 90°$.

From these relations we were able to find the forces required to move any body having elastic control, friction, or inertia, or a combination of them, with a simple harmonic motion of known amplitude and frequency. But two things were still necessary: firstly, to be able to assign a value for a force or quantity which was varying in a simple harmonic manner, and secondly, to be able to add together simple harmonic motions or forces by calculation. We therefore investigated the most suitable value to take for a periodically varying quantity, and came to the conclusion that since the work done in compressing a spring or accelerating a mass, or the power exerted in overcoming fluid friction at low speeds, was proportional to the square of the force, the effective value of such a force was such that its square was equal to the mean or average square of the force over a complete period. This effective or root-mean-square value was found to be $\frac{1}{\sqrt{2}}$ or ·707 times the maximum value in the case of a quantity varying in a simple harmonic manner.

In order to get the total result or sum of simple harmonic forces or motions, we saw that the correct method was to plot the curves of each of these forces separately, and to add these curves together. But since each of the separate motions could be represented by the vertical component of a uniformly rotating crank, we saw that the sum of the displacement due to each crank separately was given at every instant by the vertical component of a single crank, which was obtained by joining all the original cranks together, just as forces are added together to get a resultant. This led us to the conclusion that the sum of any number of simple harmonic quantities of the same frequency was a simple harmonic motion, and that this motion could always be known by putting down the cranks or vectors of the separate motions and adding them like forces. As in considering the force required to move a body, the portion required to overcome an elastic control is in phase with the displacement, while those required to overcome the friction and inertia are in phase with the velocity and acceleration respectively; and as we showed that the displacement, velocity, and acceleration for a simple harmonic motion were successively in quadrature, it followed that the vectors representing these forces were at right angles to one another, and the resultant magnitude and phase were simply found by drawing a right-angled triangle.

At the same time, since simple harmonic motions could be added like forces or vectors, it was also evident that they could be split up into components in a similar manner.

In Part III we applied the foregoing principles to alternate currents. After first showing that when a coil of wire rotates uniformly in a uniform magnetic field it must necessarily develop a simple harmonic E.M.F., we next assumed that a simple harmonic current is passed through a condenser, a resistance, or an inductance. Since in Part I we showed that quantity, current, and electrical acceleration bore the same relation to each other as the displacement, velocity, and acceleration of a body, we were enabled immediately to state, in the case of a simple harmonic current, that $\hat{Q} = \frac{1}{p}\hat{C} \overline{\diagdown} 90°$, and $\hat{A} = p\hat{C} \angle 90°$. But having also shown in Part I that the voltages required to pass a current through a condenser, resistance, or conductance were $V_1 = \frac{1}{K}Q$, $V_2 = rC$, and $V_3 = LA$ respectively, we arrived at the three fundamental alternate-current formulæ, $\hat{V}_1 = \frac{1}{K}\hat{Q} = \frac{1}{Kp}\hat{C} \overline{\diagdown} 90°$, $\hat{V}_2 = r\hat{C}$ in phase, and $\hat{V}_3 = L\hat{A} = Lp\hat{C} \angle 90°$, giving the relation between the maximum values of the P.D. and current waves if of simple harmonic form.

Next applying the result we obtained in Chapter VII for the effective value of a simple harmonic motion, *i.e.* $\frac{1}{\sqrt{2}}$ times the maximum value, we divided each of the above equations by $\sqrt{2}$, leading to the results $\bar{V}_1 = \frac{1}{Kp}\bar{C} \overline{\diagdown} 90°$, $\bar{V}_2 = r\bar{C}$ in phase, and $\bar{V}_3 = Lp\bar{C} \angle 90°$, which were the practical relations between the effective voltage and current in a condenser, resistance, or inductance simply. Inverting these equations, we had the three corresponding ones $\bar{C}_1 = Kp\bar{V} \angle 90^\circ$, $\bar{C}_2 = \frac{\bar{V}}{r}$ in phase, and $\bar{C}_3 = \frac{\bar{V}}{Lp} \ulcorner 90^\circ$, enabling the effective current to be obtained when a simple harmonic P.D. of given effective value was applied to a condenser, resistance, or inductance.

The next step was to deal with the practical cases in which resistance is combined with inductance or capacity, or all three are in series with one another. For this purpose it was only necessary to remember that when any number of circuits were joined in series, the P.D. across the combination at each instant was the sum of the voltages across each separate circuit. Hence, to get the total voltage required to pass a current through a combination of resistance, inductance, and capacity, we had only to obtain the voltage curves for each of them separately and to add the heights of these curves together to get the resultant voltage curve. But, having seen that if the current is simple harmonic, the voltages either for the condenser, resistance, or inductance are also simple harmonic, the problem became simply that of adding three simple harmonic voltages together, which, as we saw in Chapter VIII, could be done by adding their cranks or vectors like forces. We had therefore only to put down three lines representing the maximum values of the three voltages $\hat{V}_1 = \frac{1}{Kp}\hat{C}$, $\hat{V}_2 = r\hat{C}$, and $\hat{V}_3 = Lp\hat{C}$, the first lagging 90°, the second parallel to, and the third 90° in front of, the current vector, and add them together as vectors to get the maximum value of the resultant P.D. But if we divide the lengths of all the vectors by $\sqrt{2}$, we are obviously adding the effective values of the voltages together, and we obtain as our resultant the effective value of the total voltage. Finally, considering the case of passing one effective ampere only through the combination, we have $\bar{V}_1 = \frac{1}{Kp}$, $\bar{V}_2 = r$, and $\bar{V}_3 = Lp$, from which by the properties of the right-angled triangle there is immediately obtained the total effective voltage required to pass one effective ampere through the combination, or the impedance $I = \sqrt{r^2 + \left(Lp - \frac{1}{Kp}\right)^2} = \sqrt{r^2 + x^2}$ leading by an angle ϕ such that $\tan\phi = \frac{Lp - \frac{1}{Kp}}{r} = \frac{x}{r}$, where $x = Lp - \frac{1}{Kp}$ and was called the reactance of the circuit. We then had for any value of the effective current $\bar{V} = I\bar{C} \angle \phi$ or $\bar{C} = \frac{V}{I} \overline{\diagdown} \phi$.

All these things were shown to be in conformity with what should be expected from mechanical analogies, the force leading in front of the velocity in the case of swinging a dumb-bell, being in phase with it when oscillating a fan, and lagging behind it in the case of a spring. We also saw that when $Lp = \frac{1}{Kp}$, or $x = 0$, and r was small, the current became very large, and the circuit was said to be in resonance with the supply frequency. The reason was shown to be that the natural oscillation frequency of the combined inductance and capacity of the circuit coincided with that of the supply, just as the vibration of an engine or building became large when the frequency of the impulses coincided with the natural vibration frequency of the machine or building, due to its inertia and elasticity.

Coming next to circuits grouped in parallel, we saw that if we could calculate the current in each branch for a given P.D. by the rules for combinations of resistance, inductance, and capacity in series, we could find the total effective current in the mains by adding the effective values of the currents in each of the branches like vectors, exactly as voltages were added in the case of series circuits. This enabled us to find the currents for any number of circuits in parallel graphically, but in order to be able to do it by calculation recourse was had to the splitting up of simple harmonic quantities into components like forces. We thus saw that if we had a current in a branch $\bar{C} = \frac{\bar{V}}{\sqrt{r^2 + x^2}} \overline{\diagdown}\ \phi$ where $\tan\phi = \frac{x}{r}$, this could be split up into two components $\bar{C}_1 = \bar{C}\cos\phi = \frac{r}{r^2 + x^2}\bar{V}$ in phase with $\bar{V}$, and $\bar{C}_2 = \bar{C}\sin\phi = \frac{r}{r^2 + x^2}\bar{V}$ lagging 90° behind $\bar{V}$. Or if the P.D. at the terminals of the combination was one effective volt, $\bar{C}_1 = g = \frac{r}{r^2 + x^2}$, and

$\bar{C}_2 = b = \frac{x}{r^2 + x^2}$. These two quantities were called the conductance and susceptance of the branch, and were respectively the currents in phase with and lagging 90° behind the P.D. for one effective volt. As all the currents in phase with the P.D. were in phase with each other and could therefore be added together, and as the same was true of all the currents lagging 90° behind the P.D., we saw that the total conductance or current in phase with the P.D. for one effective volt is got by adding all the separate conductances together, and the total susceptance or current lagging 90° behind the P.D. for one effective volt is the sum of all the susceptances of the branches. The total current is therefore the hypothenuse of a right-angled triangle of which the total conductance and total susceptance are the two sides, or total current for one volt or admittance $Y = \sqrt{\text{conductance}^2 + \text{susceptance}^2}$ and $\tan\phi = \frac{\text{susceptance}}{\text{conductance}}$. Then $\bar{C} = Y\bar{V} \;\overline{\diagdown}\; \phi$ for any given voltage $\bar{V}$.

Our last investigation as regards a single circuit was as to the power developed or consumed. From mechanical analogy and elementary electrical knowledge we established the relation, power $w = VC$ or the product of the electrical force and electrical velocity for steady currents. We thus came to the conclusion that for a periodically varying current the mean or average power was the mean or average value of the product of the P.D. and current over the period. When the P.D. and current were in phase, this mean power in watts was obtained by multiplying the P.D. in effective volts by the current in effective amperes, so that the power was given by the product of the voltmeter and ammeter readings, just as with direct currents. When, however, the P.D. and current were simple harmonic and in quadrature, we saw, both by mechanical analogy and by symmetry, that the mean power supplied or absorbed was zero, however large the effective P.D. and current might be, although there was a pulsation of power supplied from the generator to the circuit, and from the circuit to the generator twice in each period. With a simple harmonic P.D. and current which differed in phase by an angle ϕ, the current could be split into the two components $\bar{C}_1 = \bar{C}\cos\phi$ in phase with the P.D., and $\bar{C}_2 = \bar{C}\sin\phi$ in quadrature with it, and as the power contributed by the second component is zero, the total mean power $\bar{w}$ is the product of $\bar{V}\bar{C}_1 = \bar{V}\bar{C}\cos\phi$ or $W\cos\phi$, where $W = \bar{V}\bar{C}$ is the product of the effective P.D. and current, and is called the apparent power in the circuit; the factor cos ϕ, which can only be unity or less, being called the power factor of the circuit. We also saw that the power given to an alternate-current circuit could be measured either directly by an instrument containing main and shunt coils, and known as a wattmeter, or indirectly by obtaining three voltages and constructing the vector triangle.

It is not proposed to carry this recapitulation further, as those who have thoroughly mastered the theory up to this point should have no difficulty in following the remainder. The subjects dealt with in Part IV are of great importance, and should be perfectly assimilated, but contain little new reasoning.

CONCLUSION

Before taking leave of our subject there is a question which ought to be asked and answered. We have made much use of mechanical analogies throughout our work; we have spoken of electrical friction, elasticity, and inertia; and have shown the close resemblance existing between these quantities and the mechanical quantities from which we borrowed. But how far is this legitimate? Analogies may often seem of the greatest use in assisting us over the first stages of a subject, and may yet do much harm, by giving us erroneous fundamental ideas which we have to unlearn with difficulty before proceeding to higher stages. What is therefore at the bottom of this striking resemblance between the laws of electrical motion and those of the motion of a body? Is it a trap to lure us beyond our depths, or is it something real, which is capable of being usefully employed, however far we may eventually go?

Before attempting to give any definite answer to this question let us consider the effects we have already dealt with a little more closely. The comparison between a condenser and an elastic partition in a water circuit is fairly close and seems intelligible, and the same may be said about the comparison between resistance and friction. True, we see no motion in the conductor to be retarded, and it seems difficult to think of the motion of a fluid through a solid conductor, but we are most of us familiar nowadays with the idea that all substances are composed of small particles or molecules which are quite far enough apart to allow of a fluid, or flow of other particles, passing between them. There is therefore no objection to the idea that electricity is simply an incompressible fluid, like water, moving along a circuit, except that it must differ from ordinary fluids in being able to pass along a solid metal conductor as if it were a pipe.

But when we proceed to consider inductance there are certainly one or two points which need explaining. There can be no questioning the fact that inductance controls the flow of a current, with a precisely similar law to that which expresses the controlling action of inertia upon the motion of a body. If we simply suppose that the electric fluid has ordinary mechanical mass or inertia, these effects are perfectly accounted for. Why, then, should we look further?

Well, it is perfectly simple to suppose that the electric fluid has inertia, but in that case how can we account for the fact that we can alter the inertia or inductance of a circuit over a very wide range without altering the conductor in any way? In the case of a pipe carrying a fluid we can alter the inertia in two ways only: either by altering the dimensions of the pipe so as to bring a greater or less mass of liquid into action,* or by changing the density of the liquid, as, for example, by substituting mercury for water, which would greatly increase the inertia. In the case of a moving tramcar we could load it more or less. But in all these cases we have produced some alteration *in* the moving substance itself, or *in* its channel, and nothing we can do in the space *outside* will affect the inertia. It is quite otherwise with the electric current. Changes in the section of the channel or conductor have comparatively little effect, and we know of no way of changing the nature of the fluid; but we do know that when we have a circuit of definite form, such as a spiral of a number of turns, the introduction of an iron core into the central space, which may have no contact whatever with the spiral, may increase the electrical inertia or inductance hundreds of times. There can therefore be no escape from the fact that the apparent inertia of an electric current is greatly affected by absolutely external influences; and, consequently, that if the electric current is the motion of a substance or fluid, there must be some communication or connection between it and the medium surrounding the conductor.

This single fact is crucial, and shows that there is a distinction to be drawn between the inertia of an electric circuit and that of an ordinary liquid or solid body. But it does not necessarily mean that the electric current is not the flow of a fluid, or that it is not endowed with ordinary mechanical inertia. Since we see that there is some connection between the current inside the conductor and the iron core or medium outside it, is it possible that the inertia comes from its motion being imparted to something outside the circuit which has inertia of its own and communicates it to the circuit? Now we have a good example of this in the case of a boat or other object moving through water (Fig. 186). However light such a boat may be, it always appears to have considerable inertia in view of the fact that its motion produces currents in the surrounding water, and these currents tend to persist and to maintain the motion of the boat after the driving force is removed. If we replaced the water by mercury, *i.e.* change the external medium without altering the body itself in any way, the apparent inertia may be greatly increased. Again, if we liken the electric current to the flow of water in a river, we may imagine a number of water-wheels arranged along its course so that the motion of the water is accompanied by the turning of these wheels, and it is evident that if a heavy flywheel were attached to each wheel the flow of water would be retarded at starting and assisted at stopping, just as if its own inertia had been increased.

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* The increase of the area of the pipe would not probably have any effect, as if we doubled the area there would be double the mass per unit length, but only half the velocity for the same rate of flow.

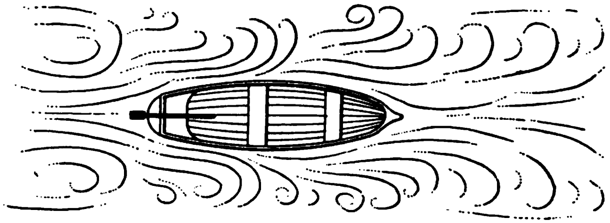


FIG. 186.—Motion of boat.

Now it will be noticed that the last idea—that of water-wheels turned by the flow of the water—corresponds very closely with the mechanism we devised to explain another property of the electric current which obviously has some relation to the medium, *i.e.* that of mutual induction between two circuits, which takes place without any mechanical contact between those circuits. If we refer back to Fig. 129 we see that we considered two adjacent circuits as if they were two cords side by side, each being coupled to a wheel, the motion of which communicated force from one circuit to the other. Now imagine, as in Fig. 187, that we had two water-pipes side by side, and that there were water-wheels between them which could somehow enter into the pipes without allowing water to escape, and most of the phenomena of self and mutual induction could be accounted for. These revolving water-wheels could, however be simply revolving eddies of water, as in the case of our boat.

If, therefore, instead of picturing a current-carrying conductor as a perfectly impervious pipe conveying water, which is absolutely isolated from the outside space, we think of it as a stream in a porous pipe immersed in water, so that the motion of the fluid in the pipe causes an eddying or flywheel motion of all the surrounding water, we shall probably be able to reconcile all the effects of self and mutual induction with those of mechanical motion. Now there is abundance of evidence from optical and electrostatic phenomena that there *is* something corresponding more or less to a fluid which pervades all space, whether vacuous or occupied by gaseous, liquid, or solid matter, and which we call ether; and we have also concluded within the last few years that this ether contains an enormous number of extremely small negatively electrified particles called electrons. What have been called the atoms of a substance now appear to be made up of a central positive mass about which we as yet know very little, with a certain number of electrons revolving around it like planets round the sun. In other words, each atom corresponds fairly closely to our pulley and mass in Fig. 130, the pulley corresponding to the rotating electrons, and the mass on which it is mounted to the central nucleus, which has a mass varying from 1700 or 1800 up to about 400,000 times the mass of an electron. We may say in a sense, therefore, that the whole of ordinary matter is full of revolving water-wheels which can gear in with the moving fluid in our conductor or pipe. The difference between a conductor and an insulator has not yet been fully accounted for, but it seems to lie in the fact that in insulators the electrons are inseparably linked with the atoms

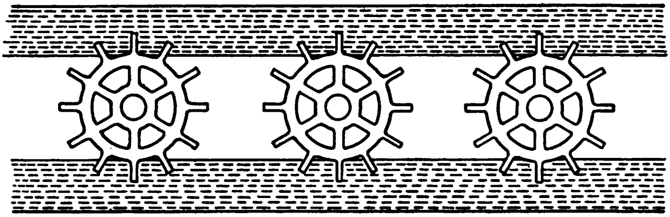


FIG. 187.—Two neighbouring water circuits.

so that they cannot move without them unless a considerable force tears them asunder, while in a conductor the electrons are handed on from one atom to another during the collisions which constantly take place between them, and can therefore flow more or less freely along the conductor under the influence of a very small directive force. In the insulator, however, although there cannot be any steady, movement of the electrons which constitutes an electric current, they can be strained over in one direction under the action of an electric force, thus producing an elastic displacement corresponding exactly to what we saw occurred in a condenser.

A further point or two about these revolving electrons will help to explain the mechanism of inductance. We know that when a current is passed round a coil of wire, the coil or solenoid has the properties of a magnet. But if the current in the coil consists of a stream of electrons, this implies that a circular stream of electrons is equivalent to a magnet. Now we have just said that each atom of matter consists of a central nucleus with electrons revolving round it, and hence this should mean that each atom is a small magnet. Those who have studied magnetism will know that it has been long recognised that a magnet is composed of a number of magnetic particles, and that each of these particles always has the same magnetic strength, the act of magnetisation consisting in facing all the little magnets round in the same direction, so that they reinforce one another. We have thus come to the conclusion that a substance consists of atoms each of which has electrons constantly rotating round and round, as in Fig. 188, and is therefore rendered magnetic, but that the act of magnetisation faces the axes of rotation round the same way, as in Fig. 189, instead of their being in all directions, as when unmagnetised. When they are revolving with their axes in the same direction, they form revolving threads or filaments, which constitute lines of force. In the ether outside the magnet the electrons are represented as whirling round, but without the central nuclei of the atoms.

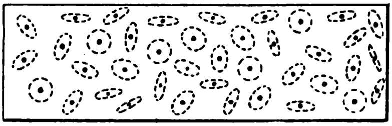


FIG. 188.—Non-magnetised bar.

According to this view we certainly have our wheels, yet the further conclusion is that they are always revolving, and that the current does not affect their rate of revolution. But what the current does is to turn each of the wheels round so that its revolving edge is along the conductor. This means that the revolving filaments or lines of force are in circles round the conductor, as we know from elementary facts. In fact the lines of force may be best likened to indiarubber rings surrounding a rod serving for the conductor, which roll round upon themselves as the rod is moved through them, or to smoke rings.

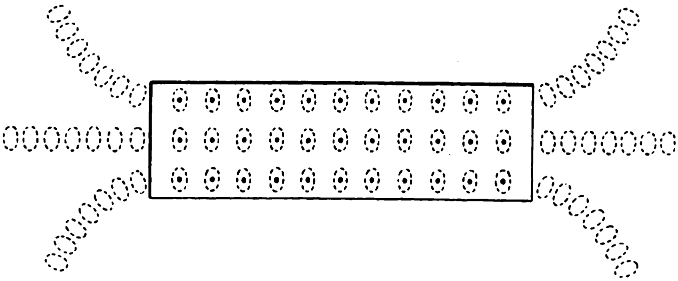


FIG. 189.—Magnetic bar with lines of force.

But it will be objected that we have now spoilt our explanation of the inertia of the circuit, by saying that the starting or stopping of the current does not set our wheels in motion, but only turns their axes round. This does not really involve any difficulty, however, as can be best shown by the aid of the gyroscope tops (Fig. 190), with which we are most of us familiar. If we take one of these tops in our hand, without spinning it, we can turn it about in any direction without effort. But if we now set the top spinning, which means that the fly-wheel is made to revolve uniformly round an axis, it will be found that any attempt to change the direction of that axis is strongly resisted. In other words, any spinning mass resists a change of the direction of its axis of rotation, as should be expected, because a change in the direction of the axis is a change of its motion, and therefore constitutes an acceleration of some sort. The starting of the current in the conductor by causing the turning of the axes of the revolving electrons is resisted just as if it had set

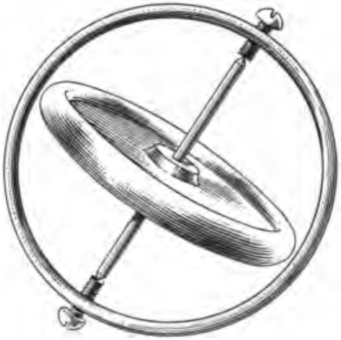


FIG. 190.—Gyroscope top.

them into motion, with a force which is proportional to the acceleration of the current, and which ceases when the current is steady; for the directions of the axes are only changed when the current is changing, and there is no force to move them back when the axes are steady.

These considerations explain exactly why the phenomena of electrical inertia are bound up with magnetism, and why an iron core which intensifies the magnetic field increases the inertia or inductance. Why the iron core should so greatly intensify the magnetic field is not fully explained; its revolving electrons must somehow have greater momentum, but whether on account of greater mass or greater velocity remains to be yet ascertained.

It will readily be seen that the phenomena of mutual inductance are perfectly accounted for by this explanation, as in the process of turning the edges of the wheels towards the conductor, the opposite edges are turned outwards towards any other conductors in the vicinity, and thus tend to drive the electrons in those conductors in the opposite direction.

Now the only other question that remains to be settled is as to how the motion of the current is caused. According to the way we looked upon the matter in Chapter I, the battery acted like a mechanical pump in a water circuit producing a total driving force or pressure. But if we consider the case of our river and paddle wheels, the motion of the water may move the wheels round, but the motion of the wheels may also move the water. Now suppose that we have an electric circuit consisting of a metal ring, and introduce a bar magnet into it as in Faraday's experiment. The result of bringing this magnet into the ring with its powerful whirls of electrons is to face all the revolving wheels within the coil in one direction, and to communicate an electrical force to the coil during the facing round process *through the wheelwork distributed over the whole length of the coil.* There can be no doubt that in all cases of currents induced by changes of magnetic flux in a circuit the force is a distributed one, as the forces of the men in a tug-of-war are distributed along the rope, or the driving force of gravity in the case of a river is distributed along its length. It has been generally supposed that this is the universal method of applying a force to the circuit, and that even when the electromotive force is provided by a battery, this force is communicated to the medium and thence to the circuit. It will be seen, however, that the method of applying the force does not in any way affect our equations or results.

One interesting conclusion from this principle may, however, be mentioned. If the driving force comes from outside through paddle wheels, and the conductor is large, the driving force is applied to the outer layers of the conductor, and the current should therefore start there first and more or less gradually penetrate to the inside of the conductor. With large conductors and alternating currents this phenomenon, known as the 'skin effect,' is found not only to exist, but to be of considerable importance, and to such an extent is this the case, that for the distribution of large alternate currents the conductors are often made in the form of tubes, as the inner portions of solid metal would be practically useless. The Deptford cable laid down by Mr. Ferranti is a good example of this.

We must now briefly sum up our conclusion in the following statements:

1. An electric current in a conductor corresponds to the flow of a fluid in a pipe, the flow consisting of electrified particles or electrons which are handed on from one atom to another. This handing on is accompanied by the collision of the atoms or molecular groups of atoms together, which checks the flow like friction, and gives rise to heat. The total or integral force along the conductor is proportional to the rate of flow.

2. An insulating substance or dielectric contains atoms with whirls of electrons which are in some way prevented from being handed on from one atom to another, probably on account of the greater complexity of the molecular groups. Under the action of an electric force, however, the electrons are strained over, thus constituting an elastic displacement or charge. An insulating partition between two conducting plates, forming a condenser, behaves, therefore, as a membranous or cellular elastic partition between two portions of a water circuit, in which the fluid in the cells cannot escape and circulate round the pipe, but can be strained over by pressure and so communicates this strain or displacement to the water on the other side. The force required is proportional to the displacement.

3. An electric current possesses inertia in virtue of the mass of its own electrons, and of the change of the directions of the axes of rotation of the electrons in the surrounding dielectric. The whirls of electrons surround a current-carrying conductor like smoke or vortex rings and constitute the lines of force, which are associated with inductive effects.

4. An electric current in a conductor is impelled by a mechanical force or difference of pressure. It is generally

supposed, however, that this force is not applied at one part of the circuit by the battery or dynamo, like the driving action of a pump in a water circuit, but is rather a force distributed over the whole length of the circuit, and communicated through the dielectric surrounding the conductor, in virtue of the rotation of its electrons, combined with the straining effect of an electrostatic field.

5. During any changes of the current or of the magnetic or electric fields, these changes are communicated from one part of the dielectric to another in all directions, at a speed equal to that of light. In the case of high-frequency electrical oscillations a considerable amount of energy may be radiated in this way, and this energy will be in the form of heat or of light if the frequency is high enough.

6. The mechanical analogies which we have made use of in alternate-current theory are thus perfectly justified, and may be extended to higher work, provided that we remember that the force arises from the medium around the conductor, and is distributed along it, and that the inertia is due to the surrounding medium as well as to the fluid or particles within the conductor itself.

EXERCISES

CHAPTER I

1. A boat is found to require a pull of 10 lbs. to tow it with a speed of 4 miles per hour. Find its friction-constant.

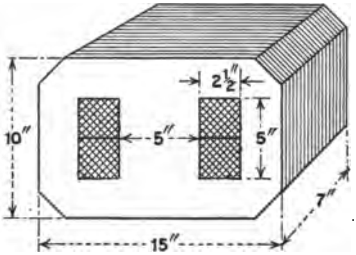
2. If the boat in the last question were towed at the rate of 4 ft. per sec., what force would be required?

3. A coil of wire was found to have a P.D. of 10 volts at its terminals when a current of 5·867 amperes was passing through it. Find its resistance, and the voltage required to pass a current of 4 amperes through it.

CHAPTER II

1. A cricket ball has a mass of 5½ ozs. Express this in engineers' units.

2. If a bowler imparts a speed of 120 ft. per sec. to the ball in a quarter of a second, find the average acceleration and the average force which he exerts on the ball.



3. Suppose that the ball moving with the velocity of 120 ft. per sec. is blocked by the batsman so that it falls without velocity, and that the time of contact with the bat is ·01 sec. What will be the average force on the bat?

4. The series coils of a generator have 10 turns, and produce a flux of 1·07 megalines when a current of 100 amperes passes through them. Find the inductance of the coils.

5. If current is switched on to the above coils and increases to 120 amperes in a quarter of a second, find the average voltage required to overcome the inductance.

6. Suppose that the current in the coils in the last question is broken by a switch which stops it in ·01 sec. What will be the average inductive voltage?

7. The following are the particulars of a 10-K.W. transformer for transforming from 2000 to 100 volts and dimensioned as in above Fig.:

Primary turns 660
Secondary turns 34

Assuming the working permeability to be $\mu = 1700$, and $7^{\circ}/_{\circ}$ of the core area to be taken up by insulation between stampings, find the total inductance of each coil in henries.

8. If the core of the transformer in the last case were divided in the centre so as to have a $\frac{1}{4}''$ air-gap, find the inductances of its coils, assuming no fringing of the flux.

9. If an E.M.F. of 100 volts were suddenly applied to one or other coil in questions 7 and 8, how would the current grow, assuming no resistance?

10. The drum armature of a 6-pole 200-K.W. 500-volt tramway generator has an inductance of ·000068 henry per coil, and a commutator having 440 parts, with a brush width of 2 segments. If the armature revolves at 135 revs. per minute, find the average reactance voltage per coil.

CHAPTER III

1. An elastic steel rope was found to stretch $1\frac{1}{2}''$ when a load of 2 tons was applied. Find its yield-constant.

2. What will be the result of stretching the rope in the last question with a force of 500 lbs.?

3. A condenser was found to pass a quantity of ·00307 coulomb when charged with a P.D. of 110 volts. Find its capacity, and the charge for a P.D. of 500 volts.

4. Calculate the capacity of a condenser composed of 100 sheets of tinfoil 1 ft. square separated by paraffined paper 4 mils thick and having a mean specific inductive capacity of 2·5.

5. A concentric cable 6 miles long has an inner conductor $1''$ diameter and a dielectric $\frac{1}{8}''$ thick, the specific inductive capacity of the dielectric being taken as 2·5. Find its capacity in microfarads.

6. If an E.M.F. of 6000 volts is switched on to the cable in the last question, find the amount of the charge.

CHAPTER IV

1. A tramcar moves at 12 miles an hour with a tractive force of 500 lbs. Find the power expended.

2. Find the kinetic energy of the car in the last question, if it has a mass of 8 tons.

3. The shunt of a 500-volt traction generator has a resistance of 284 ohms. Find the power expended in excitation.

4. A 6-pole 50-K.W. 500-volt generator has 2680 turns per pole, and the flux is 2·66 megalines per pole with a shunt current of 1·8 amps. Find the inductance of the shunt in henries, and the energy of inductance.

5. A wire rope having a yield-constant of ·005 ft. per lb. is loaded with 1 cwt. What is the work done in stretching the rope?

6. A 30-mfd. condenser is charged with a P.D. of 500 volts. What is the work done?

CHAPTER V

1. A dirigible balloon has a total weight of 4 tons and moves with a ed of 10 miles per hour, when its propellers give a propulsive

force of 200 lbs. Calculate its mass in engineers' units, and its friction-constant, assuming the force proportional to the speed. Also calculate its time-constant.

2. Plot curves showing the relation of velocity to time for the above balloon for a propelling force of 150 lbs., assuming first that it had friction without inertia, and secondly that it had inertia without friction.

3. Draw up a table as on p. 47, and plot a curve showing how the speed of the balloon increases from rest under a force of 150 lbs. with both friction and inertia.

4. Obtain the curve in the last example by the graphical method described on p. 50.

5. The exciting coil of a generator has a resistance of 13·65 ohms and an inductance of 2·78 henries. Plot curves showing the variation of current with time when a P.D. of 150 volts is suddenly applied, (*a*) when the coil is supposed to have resistance only, without inductance; (*b*) when it is supposed to have inductance without resistance; and (*c*) when both resistance and inductance are taken into account.

6. Show by Dr. Sumpner's graphical method (p. 47) how the current rises in the last exercise, and how it falls when the E.M.F. is withdrawn.

7. A 200-ton submarine has a speed of 8 knots under water when the propulsive force is 1000 lbs. If such a boat at full speed strikes a wire entanglement which yields 6″ for a force of 200 lbs., plot a curve showing its motion, neglecting the mass of the boat (1 knot = 1 nautical mile of 6280 ft. per hour).

8. In the last example find the motion of the boat on striking the entanglement, taking its mass into account.

9. A coil has a resistance of 7·28 ohms, and is connected in series with a 25-microfarad condenser. Find how the condenser becomes charged if a P.D. of 1000 volts is suddenly applied.

10. In the last question, if the coil had an inductance of 1·39 henries as well as its resistance, show what will happen on the application of the 1000-volt P.D.

CHAPTER VI

1. An engine rotates at 120 revs. per min., and the radius of its crank is 2 ft. Assuming the connecting rod to be infinitely long, draw up a table as below, and plot curves showing how the displacement, velocity, and acceleration vary with time.

Time in secs.	Angle θ	Displacement $s = 2 \sin \theta$	Difference of displacement $s_2 - s_1$	Velocity V	Difference of velocities $V_2 - V_1$	Acceleration
0						
·025						
·05						
&c.						

2. Find the amounts of the greatest displacement, velocity, and acceleration of the piston in question 1 and when they occur. Notice any other special features about the curves.

3. Describe curves of displacement, velocity, and acceleration for the above engine by the method indicated on p. 80, Chapter VI. Compare these curves with those obtained above.

4. A Willans engine has a stroke of 6″ and makes 450 revs. per min. Find the maximum displacement, velocity, and acceleration of the piston.

5. In the last question, if the piston has a mass of 150 lbs., find the maximum stress on the piston rod due to acceleration.

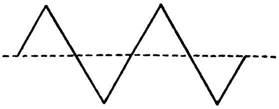
6. A punkah fan has a friction-constant of ·2 lb. for 1 ft. per sec., and is swung backwards and forwards with a stroke of 4 ft. and at the rate of 1 stroke per sec. Assuming the motion to be simple harmonic, find the maximum velocity of the fan, and the greatest force applied to it.

CHAPTER VII

1. Plot a sine curve by any method you prefer, and find the mean height of one loop of it, by measuring the area of the loop and dividing by the base. Find the relation of this mean height to the maximum height of the curve.

2. Plot a curve having the values of the angle θ in degrees as horizontal distances and of the square of the sine of the angle obtained from tables as vertical distances. Find the mean height of this curve, and the square root of this value, and compare it with the maximum height of the curve.

3. Find the effective value of a wave of V shape as shown.



4. A simple harmonic alternate current has an effective value of 150 amperes. What is its maximum value?

CHAPTER VIII

1. Draw curves representing the motion of a piston attached to a crank of 2 ft. radius revolving at 90 revs. per min., and also its motion if attached to a crank of 18″ radius lagging 70° behind the first. Add these curves together graphically, and compare the resultant curve with that generated by a single crank equal to the vector sum of the two others.

2. A fan has a mass of 5 lbs. and has a friction-constant of ·5 lb. per ft. per sec. If it is swung horizontally backwards and forwards over a stroke of 4 ft. at 80 strokes per min., find the force required.

3. If the fan in the last question is hung from the ceiling like a

pendulum and requires a force of ¾ lb. to displace it a distance of 1 ft. from its central position, find the force required to swing it in the above manner.

CHAPTER IX

1. A simple harmonic current of 10 effective amperes flows in a circuit, with a frequency of 50 ∼ per sec. Find the maximum values of the quantity, current, and electrical acceleration.

2. Find the maximum and effective values of the P.D. required to pass the current in the last questions—

(*a*) through a capacity of 50 mfd.
(*b*) through a resistance of 80 ohms.
(*c*) through an inductance of ·35 henry.

3. On p. 34 the capacity of a concentric cable 3 miles long with an inner conductor ¾″ diameter insulated by ⅛″ of vulcanised rubber was found to be 2·52 microfarads. Find its capacity reactance at 50 ∼ and the effective current which flows into the cable, when connected to an alternator giving 5000 effective volts, at that frequency.

4. The primary of a transformer contains 680 turns, and a current of ·1 ampere in it produces a total flux in the core of 1·1 megalines. Find its inductance, and the effective current which passes through it when connected to a 2200-volt 60 ∼ circuit.

CHAPTER X

1. A current of 20 effective amperes at a frequency of 50 ∼ per sec. is passed through a coil having a resistance of 5 ohms and an inductance of ·013 henry. Plot curves showing the variation of the P.D. required to overcome the resistance and inductance of the coil respectively, and thence determine the P.D. wave for the whole coil. Show that each of these curves can be represented by a crank or vector, and that the total P.D. is got by adding the vectors.

2. A coil having a resistance of 20 ohms and an inductance of ·05 henry is connected in series with a condenser of 50 mfds. capacity. If this arrangement is supplied from 1000-volt mains at a frequency of 100 ∼ per sec., find the amount and phase of the current, and the P.D. across the coil and condenser respectively.

3. A coil has a resistance of 20ω and inductance ·8 henry, in series with a 15-mfd. condenser, and is connected to a 100-volt circuit. Draw up a table as follows, showing what occurs when the frequency varies from 0 to 150 ∼ per sec.:

Frequency n	$p = 2\pi n$	Capacity reactance $x_1 = \frac{1}{Kp}$	Induc. reactance $x_2 = Lp$	Resultant reactance $X = Lp - \frac{1}{Kp}$	Impedance of coil $I_2 = \sqrt{r^2 + x_2^2}$	Impedance of circuit $I = \sqrt{R^2 + x^2}$	$C = \frac{V}{I}$	P.D. on Condenser $V_1 = \frac{1}{Kp} C$	P.D. on coil $V_2 = I_2 C$
10									
20									
30									
&c.									
|									
|									
|									
150									

Plot curves having values of the frequency n as abscissæ, and of the reactances, impedances, current, and voltages as ordinates. Note any points of interest about the curves.

4. An alternator having a resistance of 15 ohms and an inductance of ·2 henry is connected to 5 miles of concentric cable having a capacity of ·36 microfarad per mile. At what frequency would resonance take place? and what would be the current supplied to the cable and the P.D. at the alternator terminals at this frequency if the alternator E.M.F. were 2000 volts?

CHAPTER XI

1. A coil has a resistance of 5ω and an inductance of ·04 henry. Find its admittance, conductance, and susceptance; and the total, energy, and wattless current taken by it when a P.D. of 50 volts at 60 ～ is applied to its terminals.

2. If a coil of resistance 20ω and inductance ·35 henry is connected in parallel with a 50-mfd. condenser to 110-volt mains, draw up a table as on p. 136 showing what takes place as the frequency is raised from 0 to 150 ～, and plot the values of the various quantities.

3. What should be the capacity of a condenser which, when connected in parallel with a transformer having a resistance of 30ω and an inductance of 12 henries, would compensate for the wattless current. Frequency 83 ～ per sec.

4. Two coils have resistances of 5·5ω and 9·2ω, and reactances of ·033 henry and ·01 henry respectively. If these coils are connected in parallel, find the resultant resistance and reactance of the combination, for frequencies of 25 ～, 50 ～, 80 ～, and 100 ～ respectively.

CHAPTER XII

1. Find the power absorbed by a choking coil having a resistance of 5ω and an inductance of ·035 henry on 110-volt 60 ～ mains. Also find the current and the value of $\overline{C^2 r}$.

2. In the last question plot the P.D. and current curves, and the power curve obtained by multiplying the ordinates of the above two curves together. Hence find the mean power.

3. When testing an alternate-current motor a wattmeter gave a reading of 6·32 K.W., when the machine was taking a current of 45 amperes at 220 volts. Find the power factor, and angle of lag of the current.

4. A choking coil was connected in series with a non-inductive resistance, and an electrostatic voltmeter was found to read 218 volts across the mains, 156 volts across the coil, and 68 volts across the resistance, when a current of 6·5 amperes was passing through the combination. Find the power absorbed by the coil, its power factor, resistance and inductance, if the frequency of the supply was 80 ~ per sec.

CHAPTER XIII

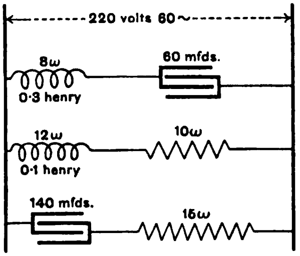
1. A circuit has an impedance of $3 + 2j$ ohms, and is traversed by a current of $5 - 7j$ amps.

(*a*) Draw to scale vectors representing the impedance and current.
(*b*) Calculate the magnitude and phase of the impedance and current.
(*c*) Calculate the magnitude and phase of the P.D. at the coil terminals.
(*d*) Check the last result graphically.
(*e*) Find, both graphically and by calculation, the power given to the coil.

2. A choking coil has a resistance of 35ω, and an inductance of ·25 henry. Express its impedance at a frequency of 50 ~ in vector form, and the current taken by it when connected to a 200-volt circuit at that frequency.

3. If the choking coil in the last question is connected in series with a 25-microfarad condenser to the same mains, find the total impedance, the current and the P.D. at the coil and condenser terminals.

4. Find the currents, P.D., and power absorbed in each of the branches of the circuit shown herewith.



5. An ammeter having a resistance r_1 and reactance x_1 is shunted with a coil having resistance r_2 and reactance x_2. Find the current in the mains corresponding to a current C in the ammeter.

6. An alternator having an armature resistance r and equivalent

inductance L supplies a current of $\tilde{C}$ effective amperes at a frequency of n periods, and lagging by an angle ϕ behind the P.D. If the alternator has a constant E.M.F. of $\tilde{E}$ volts, find how its P.D. varies with the magnitude of the current and its phase.

CHAPTER XIV

1. Two circuits having 25 and 500 turns respectively are wound side by side on a closed iron circuit having an area of 150 sq. cm., and a mean length of magnetic path of 80 cm. If the working permeability of the core is taken as 2500, find the coefficient of mutual induction between the coils.

2. If a current is started in one of the coils of the above transformer and grows steadily from zero to 5 ampères in 3 secs., find the E.M.F. induced in the other coil.

3. Find the effective voltage induced in the secondary of the above transformer when a current of 2·5 effective amperes at 60 $\sim$ is passed through the primary.

4. Two coils have total inductances of 5·7 henries and ·032 henry respectively, measured separately, and a coefficient of mutual induction of ·4 henry. Find the leakage inductances of the two coils, assuming them to have the same ratio as the total inductances.

5. In the last example, if the resistances of the two coils are 30ω and $·2\omega$ respectively, find the P.D. at the terminals of the first coil required to produce a current of 15 amperes in a non-inductive resistance of 5ω connected to the second coil.

CHAPTER XV

1. A flexible steel strip when clamped at one end is found to yield 2″ when a force of 3 lbs. is applied to it. If a mass of 5 lbs. is attached to the end of the strip, find the frequency of oscillation when disturbed.

2. A condenser having a capacity of 16·7 microfarads is discharged through a coil having an inductance of ·0155 henry. Find the frequency of oscillation.

3. A derrick crane yields 6″ under a load of a ton of bricks. At what frequency will the load vibrate?

4. If a Leyden jar having a capacity of ·006 microfarad discharges through a coil of wire having an inductance of 50 microhenries, find the frequency.

CHAPTER XVI

1. Show that in a two-phase system with three wires the current in the common conductor is $\sqrt{2}$ times that in each of the others, and the voltage between the common conductor and each of the others is $\frac{1}{\sqrt{2}}$ times the voltage between the two others.

2. A three-phase alternator has three sets of coils, the voltage in each

set being 2200, and the current 50 amperes. Write down the P.D. between lines and the current per line for this machine when the coils are connected in star or in mesh respectively.

3. A two-phase alternator gives 5000 volts per phase at 50 ∼. If one of these phases supplies current to a choking coil of 50 ohms resistance and ·5 henry inductance, and the other to a 50-microfarad condenser, find the current in each phase and the current in the common return if the supply is three-wire.

4. In a three-phase supply the P.D. between lines was 2200 effective volts, and the current 80 amperes per line. Two wattmeters connected as in Fig. 181 gave readings of 176 K.W. and 88 K.W. respectively. Find the true and apparent power supplied and the power factor and lag of the circuit if the load is balanced.

MISCELLANEOUS EXERCISES

1. Find the capacity of a mile of concentric cable from the following: Diameter of inner conductor ·5″; thickness of insulation between inner and outer conductors ·2″; specific inductive capacity of insulation 3·1.

2. An automobile weighing 3 tons and travelling with a speed of 25 miles an hour on a level road collides with an obstacle which stops it in $\frac{1}{10}$th sec. Find the average force of the impact, and the energy dissipated.

3. The shunt of a 6-pole 200-K.W. 500-volt tramway generator has 2350 turns per pole and the flux is 12·6 megalines per pole, with a shunt current of 3·25 amps. Find the inductance of the shunt in henries and the average induced voltage if the shunt is broken by a switch which breaks the current in $\frac{1}{50}$th sec.

4. A petrol engine has a stroke of 5″ and runs at 1200 revs. per min. Find the maximum piston speed and acceleration, and the stress on the piston rod due to acceleration, if the piston weighs 6 lbs.

5. An alternator having an armature resistance of ·25ω and an inductance of ·05 henry is short-circuited. If the E.M.F. of the machine is proportional to the speed and is 200 volts at 50 ∼, plot curves showing how the magnitude and phase of the short-circuit current varies with frequency, up to 200 ∼.

6. In the last question, if the machine were connected to a 100-mfd. condenser instead of being short-circuited, plot curves as before.

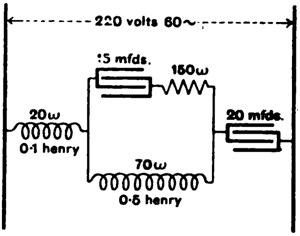
7. An automobile carrying an engine weighing 100 lbs. was found to vibrate excessively when the engine speed was 1400 revs. per min. Find the yield-constant of the frame, and state what you would do to avoid vibration if the maximum engine speed were 1600 revs. per min.

8. A circuit carries a current of 500 amps. at 83 ∼ per sec. Assuming it to vary in a simple harmonic manner, find the maximum and effective values of the quantity in coulombs, and of the electrical acceleration.

9. A current of $5 - 6j$ amperes passes through a circuit having an impedance of $4 + 3j$ ohms. Find the P.D. in volts and the true and apparent power in watts.

10. A 550-volt D.C. tramway generator has one of its terminals earthed, and the other is switched on to a cable concentric having a capacity between inner and outer conductors of 3·5 mfds., and an insulation resistance between the outer and earth of 500 megohms. Find graphically or otherwise how the cable becomes charged.

11. Find the currents, voltages, and power expended in each portion of the annexed arrangement of circuits:



12. Calculate the resistance and inductance of a coil from the following data: When connected to 200-volt 80 ~ mains the current was 5·5 amps., and when a 50ω resistance was added the currents fell to 3·2 amps.

13. An open-circuit transformer has a primary resistance of 30ω and a total primary inductance of 13·5 henries. Find the no-load current when the primary is connected to a 2000-volt 50 ~ circuit, and the capacity of a condenser which would compensate for the wattless current. Also find the value of the current in the mains when the condenser is connected.

14. Find the E.M.F. of an alternator having a terminal P.D. of 2200 volts when supplying a current of 1000 amps. lagging by 60°, the armature resistance being ·174ω and the reactance ·76ω.

15. Two coils have resistances of 2·4 and 7·5ω respectively, leakage inductances of ·01 and ·02 henry, and a coefficient of mutual inductance of ·3 henry. Draw a vector diagram showing what occurs when a current of 2 amps. at 100 volts and a power factor of ·7 is taken from the second coil.

16. A Leyden jar having a capacity of ·008 mfd. is discharged through a circuit having an inductance of 60 microhenries. Find the frequency of the oscillations produced.

ANSWERS TO EXERCISES

CHAPTER I

(1) 1·705 lbs. (2) 6·82 lbs. (3) 1·705 ohms, 6·82 volts.

CHAPTER II

(1) ·01068. (2) 480 ft. per sec. per sec., 5·125 lbs. (3) 128·4 lbs. (4) ·00107 henries. (5) ·5136 volts. (6) 12·84 lbs. (7) Primary Inductance, 30·7 henries; Secondary Inductance, ·0815 henry. (8) Primary Induct., 1·7 henries; Secondary Induct., ·00451 henry. (9) In (7) Prim. = 32·6 amps. per sec; Sec. = 1227 amps. per sec. In (8) Prim. = 58·8 amps. per sec.; Sec. = 22,160 amps. per sec. (10) 4·48 volts.

CHAPTER III

(1) ·0000279. (2) ·01395 inch. (3) ·0000279 farad, ·01395 coulomb. (4) 4·01 microfarads. (5) 6·015 mfds. (6) 36,090 microcoulombs.

CHAPTER IV

(1) 16 h.p. (2) 38·4 ft.-tons. (3) 880 watts. (4) 236 henries, 382 joules. (5) 62·72 ft.-lbs. (6) 3·75 joules.

CHAPTER V

(1) $m = 278, f = 13{\cdot}64, T = 20{\cdot}4$.

CHAPTER VI

(2) Greatest displacement, 2 ft. at ends of stroke; greatest velocity, 25·12 ft. per sec. at middle of stroke; greatest acceleration, 316 ft. per sec. at opposite ends of stroke to displacement. (4) 6 inches; 23·55 ft. per sec.; 1110 ft. per sec. per sec. (5) 74·4 tons. (6) 12·56 ft. per sec.; 2·512 lbs.

CHAPTER VII

(1) 0·637. (2) ·5000, ·7071. (4) 212·13 amps.

CHAPTER VIII

(2) 23·35 lbs. (3) 6·84 lbs. (4) 8·5 lbs.

CHAPTER IX

(1) $\hat{Q} = {\cdot}045$ coulomb; $\hat{C} = 14{\cdot}14$ amps.; $\hat{A} = 434{\cdot}5$ per sec. (2) (*a*) 900 volts; 637 volts; (*b*) 1130 volts; 800 volts; (*c*) 155 volts; 1099 volts. (3) 1265 ohms; 3·955 amps. (4) 74·8 henries; 4·145 amps.

CHAPTER X

(2) 50 amps. approx. leading 1° 17′; coil P.D. 1860 volts, condenser P.D. 1590. (4) 153·5 ∼ per sec.

CHAPTER XI

(1) $Y = {\cdot}0626$; total, $C = 3{\cdot}13$ amps. $g = {\cdot}01985$; energy $C = {\cdot}09925$. $b = {\cdot}0595$; wattless $C = 2{\cdot}975$. (2) 1910 mfds. (4) ∼ 25 resultant $r = 3{\cdot}76$ ohms, resultant $x = 2{\cdot}02$ ohms; ∼ 50 resultant $r = 4{\cdot}48$ ohms, resultant $x = 3{\cdot}57$ ohms; ∼ 80 resultant $r = 5{\cdot}02$ ohms, resultant $x = 4{\cdot}91$ ohms; ∼ 100 resultant $r = 5{\cdot}22$ ohms, resultant $x = 5{\cdot}79$ ohms.

CHAPTER XII

(1) 305 watts, 7·81 amps., C^2r loss 305 watts. (3) 0·639; angle of lag 50° approx. (4) 975 watts, 0·963, 23·1 ohms, 14·3 millihenries.

CHAPTER XIII

(1) (*b*) 3·61 ohms, 8·6 amperes; (*c*) 31 volts; (*e*) 68 watts.

(2) $\mathrm{I} = 35 + 78{\cdot}54\,j = 85{\cdot}9 \;\angle\, 65^\circ\, 56'$ ohms.

$C = 0{\cdot}946 - 2{\cdot}125\,j = 2{\cdot}33 \;\ulcorner\, 65^\circ\, 56'$ amps.

(3) $\mathrm{I} = 35 - 49\,j = 60{\cdot}2 \;\angle\, 54^\circ\, 30'$ ohms.

$C = 1{\cdot}93 + 2{\cdot}7\,j = 3{\cdot}32 \;\angle\, 54^\circ\, 30'$ amps.

Coil P.D. $= 144{\cdot}5 + 245{\cdot}8\,j = 285 \;\angle\, 120^\circ\, 26'$ volts.

Condenser P.D. $= 344{\cdot}5 - 245{\cdot}8\,j = 423 \;\ulcorner\, 35^\circ\, 30'$ volts.

(4) Upper Circuit Current = 3·17 amps., condenser P.D. = 140 volts, watts = 0, coil P.D. = 359 volts, watts = 80·5. Middle Circuit Current = 5·05 amps., coil P.D. = 200 volts, watts = 306, resistance P.D. across = 50·5, watts = 255. Lower Circuit Current = 9·275 amps., condenser

P.D. = 171 volts, watts = 0, resistance P.D. across = 139·3 volts, watts = 1290 watts.

(6) Main Current $= C\sqrt{\dfrac{(r_1+r_2)^2+(x_1+x_2)^2}{r_2^2+x_2^2}}$

$$\text{P.D.} = \sqrt{E^2 - C^2\,(r\sin\phi + x\cos\phi)_2} - C\,(r\cos\phi - x\sin\phi)$$

where $x = Lp = 2\pi nL$.

CHAPTER XIV

(1) 0·736 henry. (2) 1·225 volts. (3) 694 volts. (4) ·36 henry, ·002 henry. (5) 1110 volts.

CHAPTER XV

(1) 1·7 ~ per sec. (2) 313 ~ per sec. (3) 1·28 ~ per sec. (4) 290,000 ~ per sec.

CHAPTER XVI

(2) Star, 3810 volts, 50 amps.; Mesh, 2200 volts, 86·6 amps. (3) 30·3 amps., 78·7 amps., 92·5 amps. (4) True power, 264 kilowatts; apparent, 305 kilowatts; power factor, ·866; angle of lag, 30°.

MISCELLANEOUS EXERCISES

(1) 0·825 mfds. (2) 34·2 tons, 62·5 ft.-tons. (3) 546·6 henries, 88,800 volts. (4) $\hat{v} = 54{\cdot}2$ ft. per sec.; $\ddot{a} = 6570$ ft. per sec. per sec.; $F = 1224$ lbs. (7) $Y = 15{\cdot}7 \times 10^{-4}$; stiffen frame till Y is less than 12×10^{-4}. (8) $\bar{Q} = 0{\cdot}958$ coulomb. $\hat{Q} = 1{\cdot}352$ coulombs. $\bar{A} = 261{,}000$ amps. per sec. $\hat{A} = 368{,}000$. (9) P.D. 39·05 volts; true watts, 244; app. watts, 366. (11) (A) ·9825 amps., 41·4 volts, 19·3 watts. (B) 205 volts, current = ·885 amps., condenser P.D. = 156·5 volts, con. watts 0, res. P.D. = 132·5 volts, res. watts = 117·2 watts. (C) 205 volts, $C = 1{\cdot}018$ amps., watts 72·5. (D) $C = {\cdot}9825$ amp., $V = 130$ volts, watts 0. (12) $r = 0{\cdot}8$ ohm, $L = 72{\cdot}5$ millihenries. (13) 0·472 amp., 0·75 mfds., 3·3 milliamps. (14) 2945 volts approx. (16) 230,000 per sec.

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